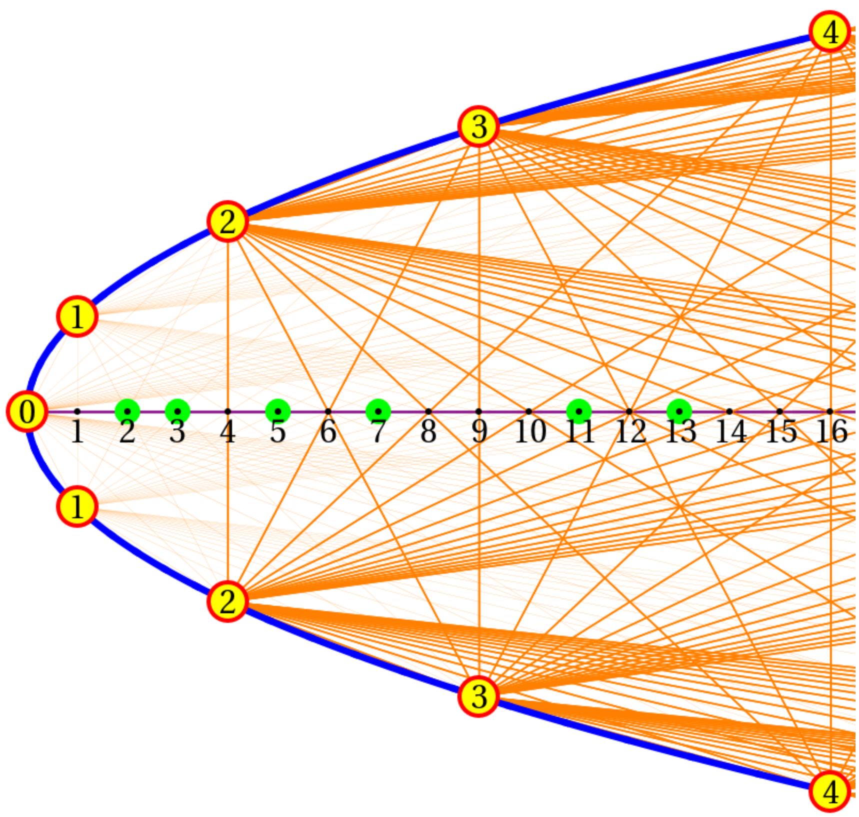
Magic with a Parabola

Objective

In this first lecture, we look at a mysterious property of the parabola. The parabola allows to multiply and divide numbers geometrically and even find primes. Why? We are going to find out!

This "mystery result" of this first class introduces us to a few topics in mathematics, like the nature of **arithmetics**, prime numbers in **number theory**, **group theory**, **algebra**, symmetries in **geometry**. It involves even the concept of limits and tangents which are used in **calculus**.

I learned this multiplication from a video by **George Hart** from the opening of the MoMath museum in New York. I later found the parabola also in the picture Book **"Mathematics, 100 Breakthroughs that changed History", 2012** which is like Pickover's **"The Math book"** a good companion for this course. They are both very much in the same "case style" like our course and best advertisement for the beauty of mathematics.

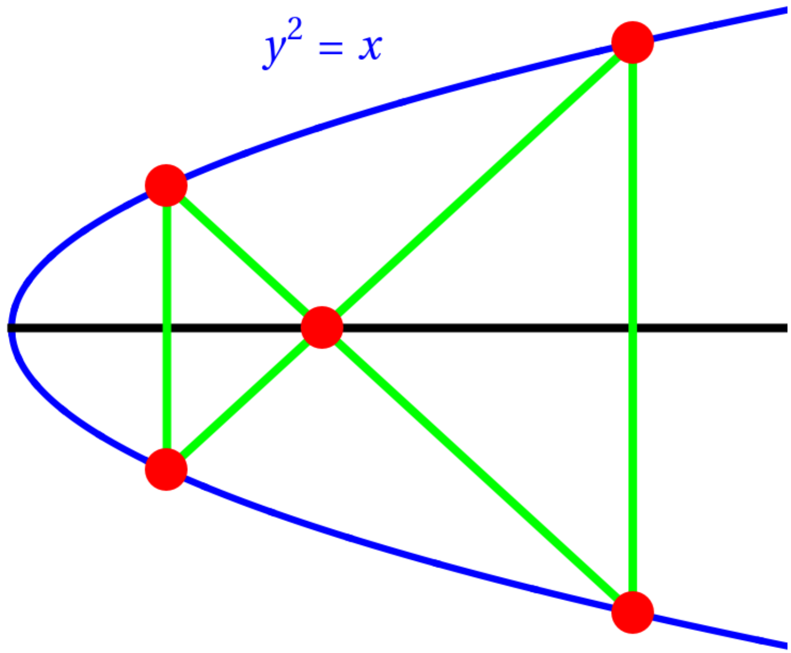


The multiplication

Connect the point 2 with 4 and look at the point of intersection with the horizontal axes. Do this with other numbers. At the end of this handout is a larger parabola, where you can experiment. You will see there the parabola turned and negative numbers on the lower branch. This will allow us to multiply also negative numbers.

Formulating the result

When connecting two numbers on the parabola, the intersection with the symmetry line gives the product.

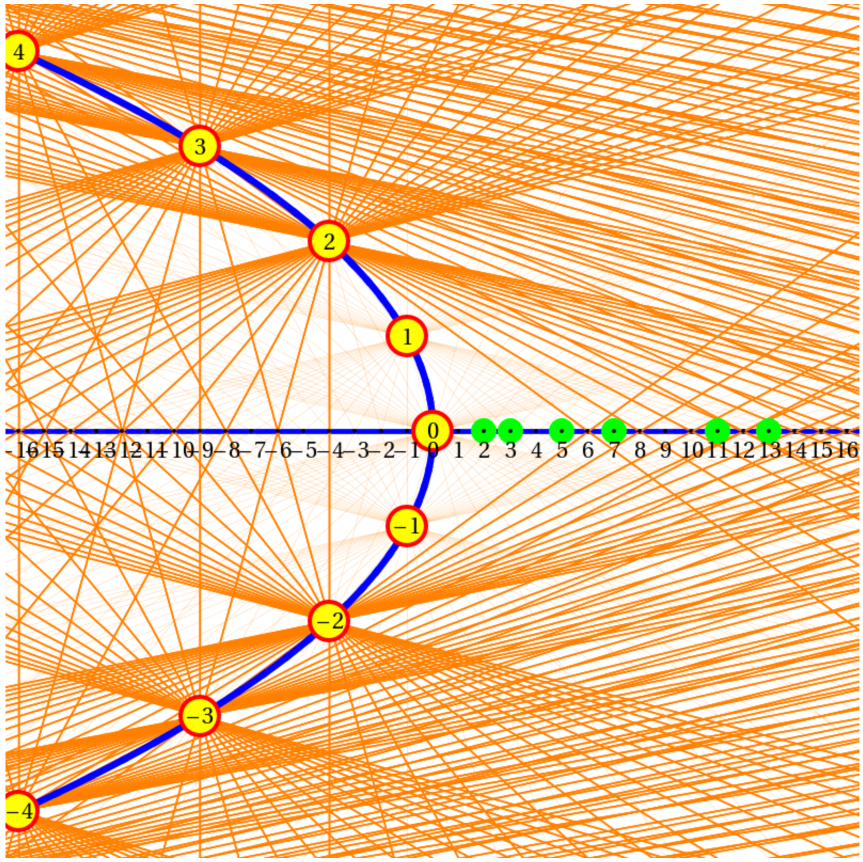


Proving the result

Look at the two similar triangles and work from there. Their relation is x/y. Can you see where the diagonals intersect? This needs a bit of geometry and algebra. We will work on it together and see whether we can make a better 6 second movie together.

The group of integers

We can put the real axes onto the parabola. We have now also negative numbers. We see $y^2 = -x$. Now, we can multiply any two real numbers geometrically.



How to find primes

Primes are integers larger than 1 which can only be divided by 1 or itself. The parabola implements the **Sieve of Erastostenes** for finding primes. How?

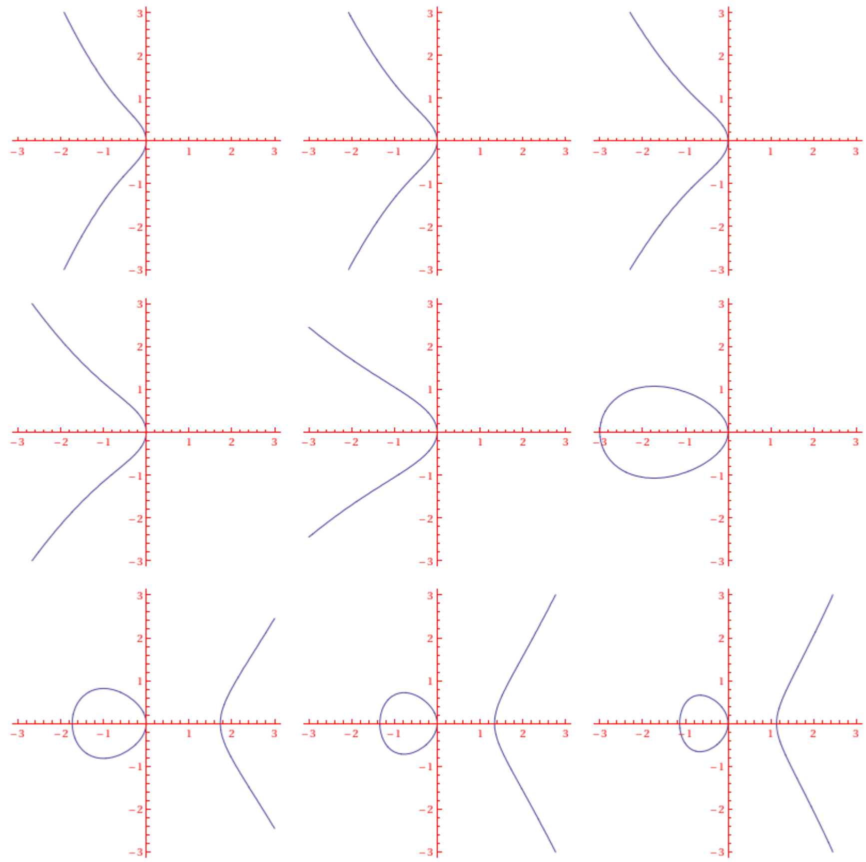
Relation with other mathematics

The relation of group theory and geometry goes much further. Instead of parabola, we can take find a multiplication on **elliptic curves** like

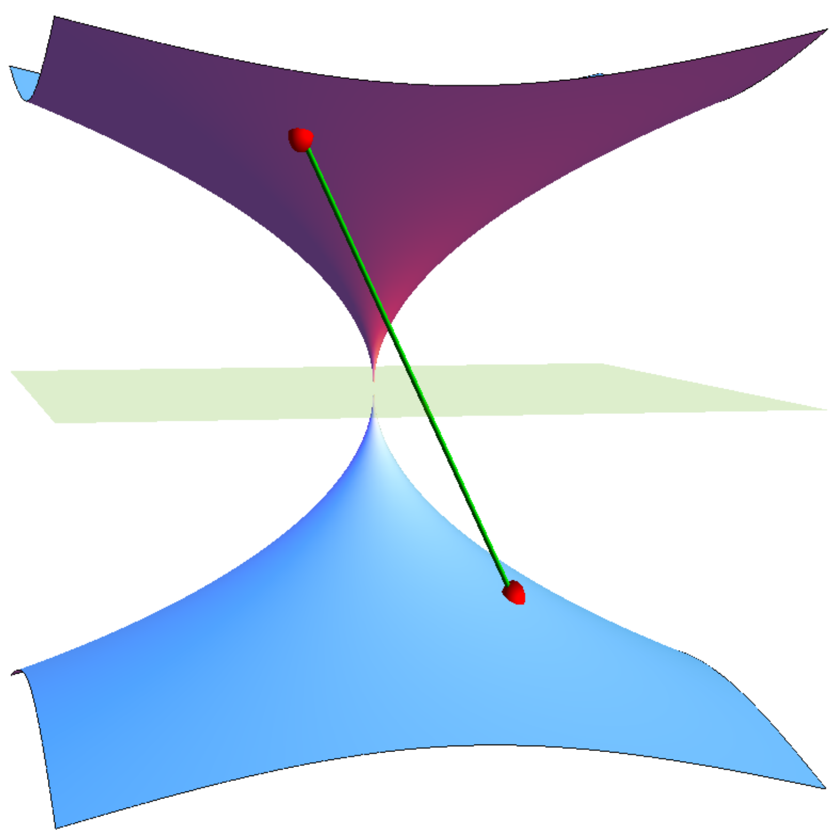
$$y^2 = ax^3 - x \ .$$

You see a few pictures of this curve below. These objects are **very** important in number theory. The multiplication on elliptic curves is useful in **cryptology**. In some sense, we can continue the multiplication to the case $a = 0$, when the elliptic curve has degenerated to a parabola.

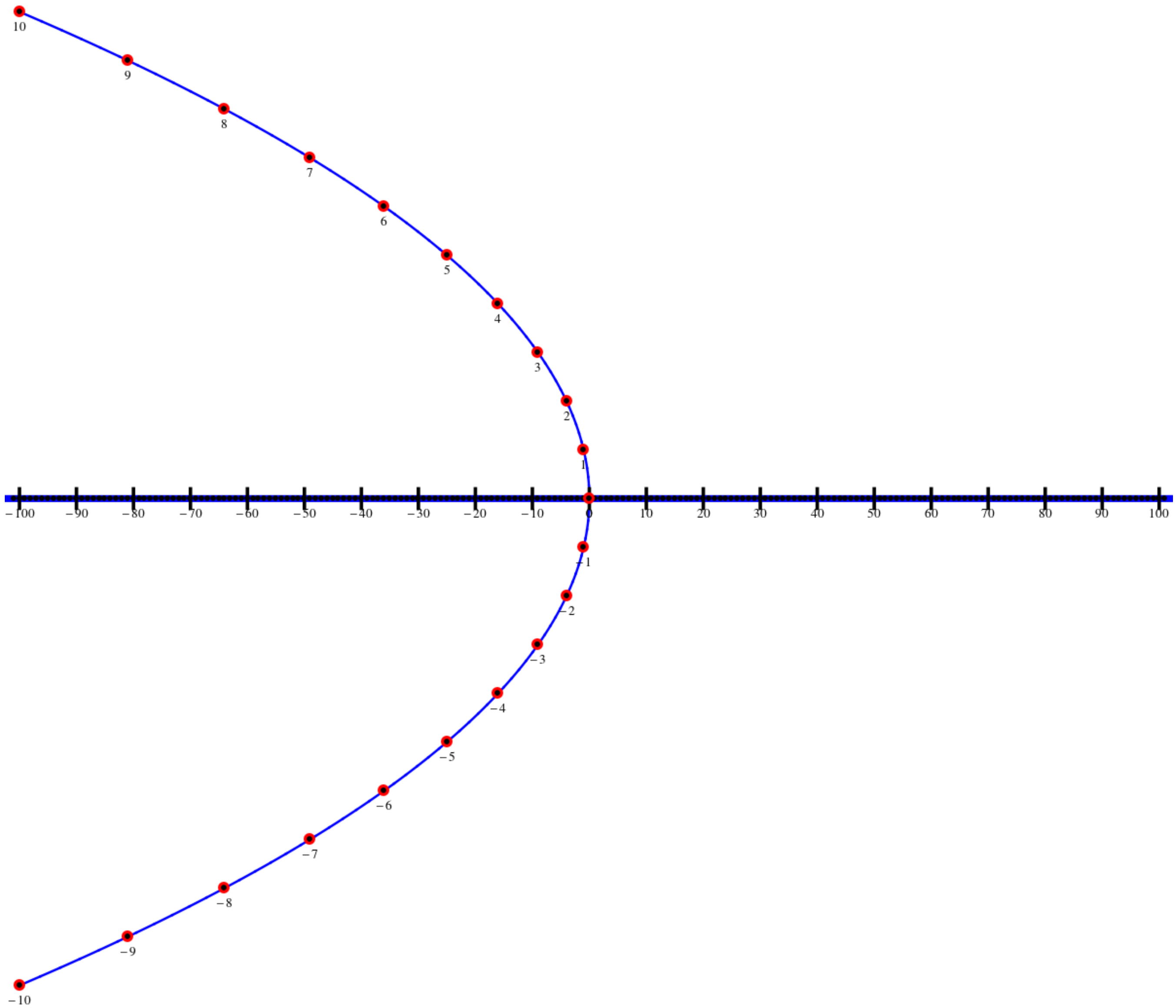
I do not know yet, whether the elliptic curve addition naturally goes over to the parabola multiplication treated here. The process is suspiciously similar. In both cases, one connects two points with a line, intersects with the curve and then then maps the intersection point back onto the curve with an involution.



This multiplication which we have just seen for real numbers can be done also with complex numbers. This can no more been drawn as a graph, since we would have to do that in 4 dimensions. But in principle, everything is the same. Every complex number z leads to two complex numbers $\pm\sqrt{z}$. The graph is called a "Riemann surface". Now take two points z, w on that surface and connect them and intersect the line in C^2 with the complex plane $C \times 0$. This result is the multiplication $z \star w$.

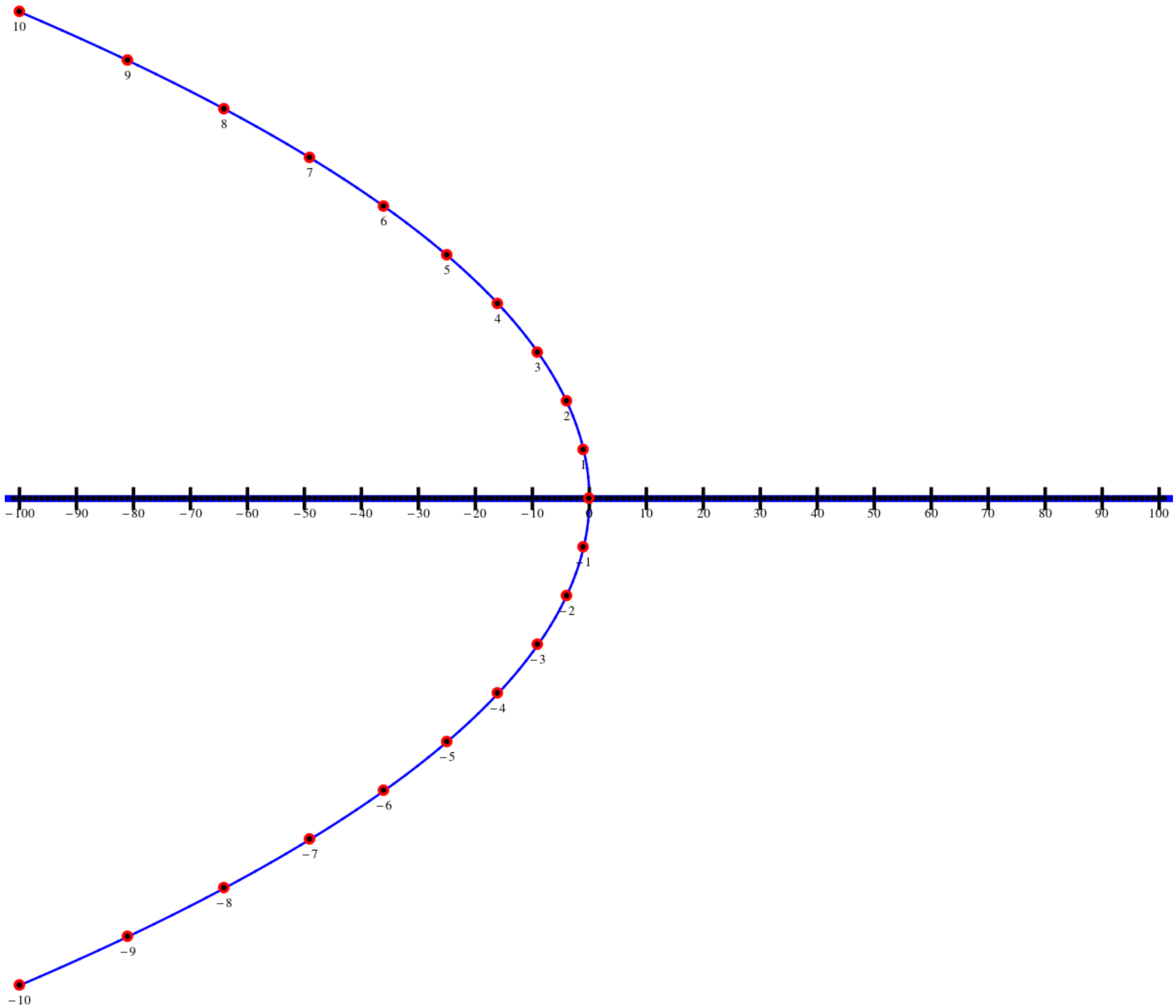


Sketchbook.



Use the included parabola to make some computations. Make sure to try out also cases $x \cdot y$ where x, y are the same. Does it work also for $x = 0$? Can you see how the limiting process works? Can you find a way to divide numbers?

More Sketchbook.



Try out also the case $x \cdot (-x)$, which is quite obvious. We can deduce geometric results about the parabola.