

## Roots

We start with a theorem discovered by **Hippasus of Metapontum** from 500 BC. Legend tells that the discoverer had to pay with his life for the discovery of incommensurable magnitudes:

### $\sqrt{2}$ is irrational

Assume  $\sqrt{2} = p/q$ , then  $q\sqrt{2} = p$  and  $2q^2 = p^2$ . Since the number of factors 2 on the left are odd and even on the right, this is a contradiction.

This works for any  $\sqrt{n}$  as long as  $n$  is not a square. **Theodorus of Cyrene**, a contemporary of Hippasus who extended some irrationality proofs as we know from his students **Theatetus of Cyrene** and **Plato**. The just given irrationality proof relies on the **fundamental theorem of arithmetic** proven by the 21 year old **Karl-Friedrich Gauss** in 1798 in his *Disquisitiones Arithmeticae*. In our slides we show a geometric proof by descent which does not need the unique prime factorization result.

## Logarithms

Logarithms were introduced independently by the English mathematician **John Napier** 1550-1617 and the Swiss mathematician **Joost Bürgi** 1552-1632. Logarithms invert the exponential. For example,  $\log_{10}(1000) = 3$  because  $10^3 = 1000$ . An other example is  $\log_2(1/64) = -6$  as  $2^{-6} = 1/64$  or  $\log_{10}(\sqrt{10}) = 1/2$ .

### $\log_{10}(2)$ is irrational

If  $\log(2) = p/q$ , then  $2 = 10^{p/q}$  and so  $2^q = 10^p$ . The right hand side is divisible by 5, the left not.

**Question:** Is  $\log_4(2)$  rational or irrational? **Answer:** it is rational. Can you see why?

**Open problem:** Is there an irrational  $x$  such that  $2^x$  and  $3^x$  are integers?

This would mean that there exist two integers  $n, m$  such that  $\log_2(n) = \log_3(m)$ . We could try to search with a computer by hunting down integer pairs  $n, m > 1$  but it could be a futile hunt because most likely there does not exist such a pair.

## Powers

One knows that  $\pi$  and the Euler constant  $e$  are irrational. We will look at the proof that  $\pi$  is irrational in a moment. The number  $G = e^\pi = 23.14\dots$  is called **Gelfond's constant**. Hilbert had asked in his 7th problem whether this number is rational. Since  $-1 = e^{i\pi}$ , we have  $G = (-1)^{(-i)}$ . It follows from a theorem of Gelfond-Schneider that  $G$  is irrational as  $(-1)$  is rational and  $-i$  is algebraic. Also the **Gelfond-Schneider constant**  $\sqrt{2}^{\sqrt{2}}$  is known to be irrational. We will come back to this. The theorem of Gelfond-Schneider is beyond the scope of this course. It is part of transcendental number theory. **Alexander Gelfond** (1906-1968) was a Russian mathematician who was the chief cryptograph of the Soviet Navy during WW II. **Theodor Schneider** (1911-1988) was a German mathematician who was a student of **Carl Ludwig Siegel**. The later was also the doctoral advisor of **Jürgen Moser**, who was my own undergraduate thesis advisor. Schneider served around 1960 as the director of the Oberwolfach research institute in the "Schwarzwald".

## $i^i$ is irrational

$i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 1/\sqrt{e^\pi} = 1/\sqrt{G} = 0.20788\dots$  Assume this is  $p/q$ , then  $1/G = p^2/q^2$  and  $G = q^2/p^2$ . But that would imply that the Gelfond's constant  $G$  is rational which we know is not true.

So, we have seen that the "eye for an eye" number is irrational. Quite a metaphor. Here are more problems for which the answers are unknown:

**Open problem:** Is  $\pi^e$  irrational?

**Open problem:** Is  $\pi^{(\pi^{(\pi^\pi)})}$  an integer?

For an ultrafinitist (which many computer scientists are by nature), a number like  $e^{(e^{(e^e)})}$  does not exist yet as we can not realize it yet in a computer. Note the order of the brackets. While we have no idea whether  $x^{(x^{(x^x)})}$  is an integer if  $x = \sqrt{10}$  (it is inaccessible to us), we know that  $((x^x)^x)^x$  is an integer as it is  $\sqrt{10}^{100} = 10^{50}$ .

## Logic

I learned the following result as a student from **Gerhard Jäger** who is now in the logic and theory group at the University of Bern in Switzerland. The proof is attributed to **Dov Jarden**.

**There exist irrational  $x, y$  such that  $x^y$  is rational.**

There are two possibilities. Either  $z = \sqrt{2}^{\sqrt{2}}$  is irrational or not. In the first case, we have found an example where  $x = y = \sqrt{2}$ . In the second case, take  $x = z$  and take  $y = \sqrt{2}$ . Now  $x^y = \sqrt{2}^2 = 2$  is rational and we have an example.

In this case, we know by Gelfond-Schneider's theorem that the first case happens. A related result by Ash and Tan proves that

## There exists an irrational $x$ such that $x^x$ is rational.

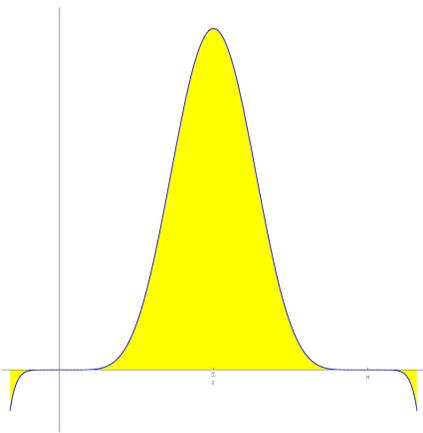
The function  $f(x) = x^x$  maps  $(1/e, \infty)$  bijectively to  $(f(x), \infty)$ . The case that  $x$  and  $f(x)$  are both rational is extremely rare. It only happens if  $(x, f(x))$  is an integer pair of the form  $(n, n^n)$ : assume  $x = p/q$  and  $f(x) = a/b$  are reduced fractions, then  $(p/q)^p = (a/b)^q$  or  $p^p b^q = a^q q^p$  this implies that a prime divides  $b^q$  if and only if it divides  $q^p$  so that  $b^q = q^p$ . This implies  $q = b = 1$ . Having shown that the pair  $x, x^x$  both being rational only occurs if  $x$  is an integer shows that we can solve for example  $x^x = 7/5$  for  $x$  and know that  $x$  must be irrational.

## Pi

Babylonian mathematicians knew that  $\pi$  is close to  $25/8$ . **Archimedes** proved around 300 BC that  $223/71 < \pi < 22/7$  and Ptolemy used in 200 AC the approximation value  $377/120$  which is off by  $0.000074\dots$  only. Today, 12.1 trillion digits of  $\pi$  are known. A trillion digits corresponds to a 1 TB hard drive. When compressed, one could probably still fit the known digits on a 1 TB hard drive. The Swiss mathematician **Johann Lambert** proved in 1761 that  $\pi$  is irrational. We follow a proof of the Canadian mathematician **Ivan Niven** from 1946 which uses the **fundamental theorem of calculus**.

## $\pi$ is irrational.

Assume  $\pi = p/q$ . Take a large  $n$ . Define  $f(x) = x^n(p - qx)^n/n!$ , a function whose graph we see below for  $n = 7$ . The function  $F = f - f^{(2)} + f^{(4)} - \dots$  has the property that  $F(0), F(\pi)$  are both integers and that  $(F' \sin(x) - F \cos(x))' = f(x) \sin(x)$ . Integrating this gives  $\int_0^\pi f(x) \sin(x) dx = F(\pi) - F(0)$  which is a positive integer as  $f(x) \sin(x)$  is positive between 0 and  $\pi$ . On the other hand,  $f(x) \sin(x) < \pi^n p^n/n!$  is arbitrarily small for large  $n$ . This contradiction shows  $\pi = p/q$  is absurd.



**Remark:** there is a physics connection: write  $D$  for the derivative. The equation  $(1 + D^2)F = F + F'' = f$  is a **driven harmonic oscillator** where  $f$  is a **time dependent force**. It is solved by  $F = (1 - D^2 + D^4 - D^6 \dots)f$ , which is a finite sum as  $f$  is a polynomial. The rationality of  $\pi$  implies that  $f(0) = f(\pi) = 0$  and  $f^{(2k)}$  are all integers. The fact that  $F(\pi) - F(0)$  is a positive integer means that the oscillator's amplitude has grown by an integer from time  $x = 0$  to  $x = \pi$ . But the rationality of  $\pi$  implied the existence of forces  $f$  with arbitrary small  $\int_0^\pi f(x) \sin(x) dx$ . For large  $n$ , there is not enough "energy" to be pumped into the system to lift the oscillator amplitude by a positive integer.