Chaotic evolution and strange attractors

The statistical analysis of time series for deterministic nonlinear systems

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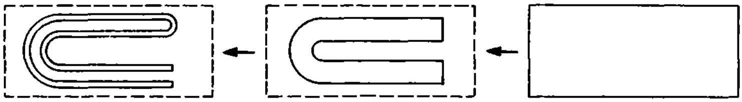
The Hénon mapping

Consider the discrete time evolution equation:

$$x(t+1)=y(t)+1-a[x(t)]^2$$
$$y(t+1)=bx(t). \qquad (17)$$

Hénon has found this equation by looking for a system which is as simple as possible, yet exhibits the same essential properties as the Lorenz system for certain values of the parameters of that system (not those of Fig. 5) (see Hénon, 1976). The motivation for this research was to provide a more handy model for numerical explorations. The fact that the mapping (17) is discrete in time and two-dimensional, while the Lorenz system is continuous in time and three-dimensional, is because (17) is obtained as a first return map for a surface of section; namely, it is a mapping obtained considering the successive intersections of the three-dimensional flow with a surface Σ of codimension one and transversal to the direction of the flow. Such a mapping is often called Poincaré map. A trajectory is thus replaced by a discrete set of points in Σ, and all the essential properties of the trajectory are carried into corresponding properties of this set of points in such a way that one can forget about the differential system and focus the attention on the mapping T of Σ into itself. Then one can define an explicit equation giving directly $T(x_0,y_0)$ when the initial point with coordinates (x_0,y_0) on Σ is known. Eq. (17) is obtained in such a way and simulates the stretching in one direction and folding over which is a typical behavior with (9). Note that the Hénon dynamics (Fig. 6) is

Fig. 6. The Hénon dynamics: contracting of volume, stretching of length and folding.



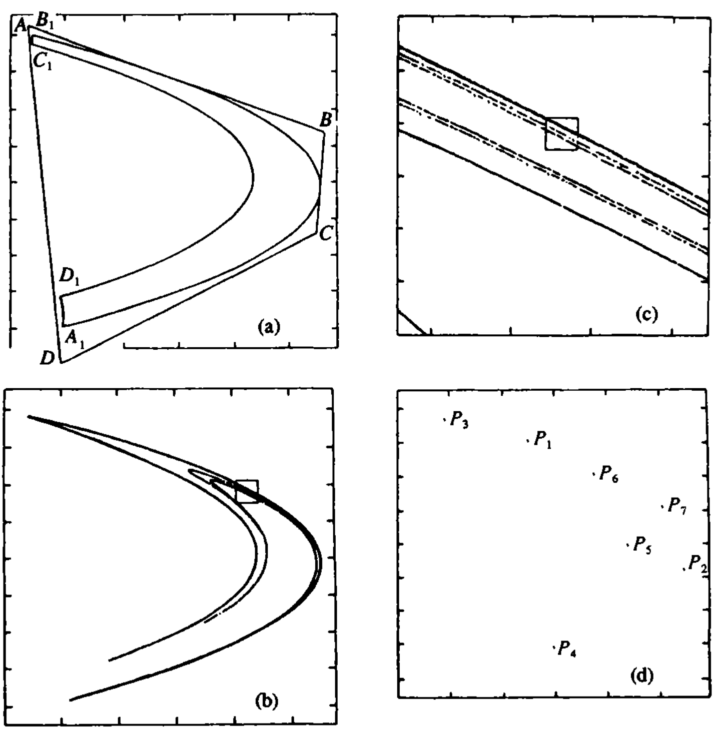
different from Smale's horseshoe dynamics (see Smale, 1967) which does not correspond to an attractor.

Let us notice that the mapping (17) has a constant jacobian:

$$J = \frac{\partial(x(t+1),y(t+1))}{\partial(x(t)),y(t))} = -b. \tag{18}$$

The geometrical interpretation of (18) is that the action of (17) contracts areas by a factor b and reverses the orientation if $0<b<1$ (note that an honest Poincaré map would preserve orientation, so that the Hénon map for the usual values $a=1.4$, $b=0.3$ cannot really be interpreted as Poincaré map!) We can observe this behavior in Fig. 7a, where the quadrilateral $ABCD$ (which is called the trapping

Fig. 7. The Hénon attractor (see text for details). From Curry (1979) and Hénon (1976).



region) is mapped into $A_1B_1C_1D_1$ by the diffeomorphic map of the plane into itself:

$$(x,y)\to(y+1-ax^2,bx). \tag{19}$$

Fig. 7b shows the result of a numerical experiment where an initial point is evolved iterating (17) 10000 times, with $a=1.4$ and $b=0.3$. We can see that the successive points distribute themselves on a complex system of lines which is entirely contained in the trapping region $ABCD$ of Fig. 7a. A magnification of the little square in Fig. 7b yields Fig. 7c, and a further magnification would yield again a similar picture. This system of lines constitutes a strange attractor. The self-similarity visible in Fig. 7b, c is a typical property of a set which is invariant under time evolution, and makes it a *fractal* set (see Mandelbrot, 1982). In the case of the Hénon attractor this is due to its transversal structure (across the lines), which is Cantor-like. If we take $a=1.3$, $b=0.3$, the strange attractor of Fig. 7b is replaced by the attracting periodic orbit (with period 7) of Fig. 7d.