Lecture 12: Dynamical systems

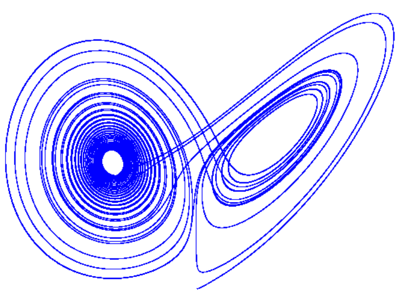
Dynamics

Dynamical systems theory studies time evolution of systems.
If time is **continuous** the evolution is defined by a **differential equation** $\dot{x} = f(x)$. If time is **discrete** we look at the **iteration of a map** $x \to T(x)$. The goal is to **predict the future** of the system when the present state is known.

Here is the prototype of a differential equation

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \ . \end{aligned}$$

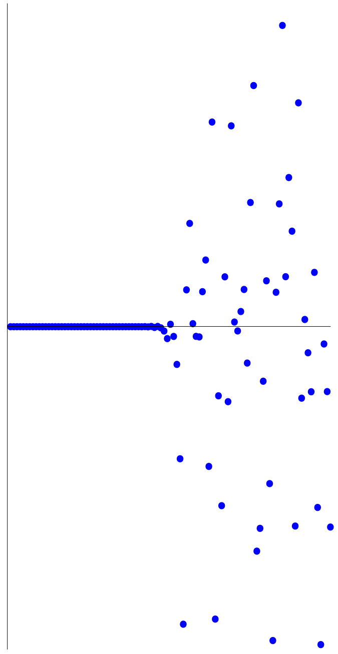
the **Lorenz system**. There are three parameters. For $\sigma = 10, r = 28, b = 8/3$, one observes a **strange attractor**.



Chaos

There are various definitions of chaos. One of them is "sensitive dependence on initial conditions". The smallest change of the initial state produce large changes in the future. We illustrate this with the example of the logistic, where already for $n = 60$, there is a big difference between the orbit of $T(x) = 4x(1 - x)$ and $S(x) = 4x - 4x^2$.

The following picture shows the difference of $T^n(x) - S^n(x)$ for $n = 0$ to 100 starting with 0.3. With 16 digits of accuracy of computation it does not make sense to talk about the value of $T^{100}(0.3)$. It is undefined evenso the map is very concrete and deterministic. Increasing the accuracy does not help much. The error doubles in each step so that we expect to see an error of the order 1 after $\log_2(10^{16}) = 53$ steps. If we computed with 100 digits accuracy, then we would have lost all information after $\log_2(10^{100}) = 332$ steps. Because $\log_2(10) = 3.32193$ we have only have to iterate 3.4 times the given accuracy to have no idea where the $n'th$ iterate is. As we can see, the value depends then for example on how we have written down the equations.



Already simple maps produce, when iterated unpredictable results.