Lecture 7: Worksheets

1. Objective

There are two algebraic operations $+$ and $\cdot$ on the set of all subsets. We can visualize them graphically with **Venn diagrams**.
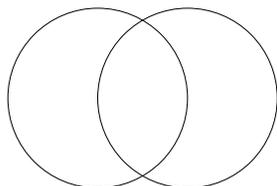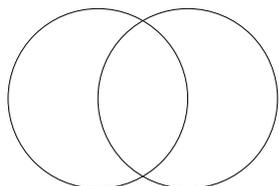
2. The Boolean ring

Addition
$A + B = A \Delta B$
$0 = \emptyset$

Multiplication
$A \cdot B = A \cap B$
$1 = X$.

1) How can we write the union of two sets $A \cup B$ using addition and multiplication.

2) Verify that $A = -A$. The set A is the unique set which satisfies $A + A = \emptyset$.

3) Draw the Venn diagram of $A - B = A + (-B)$.

4) How can we write the set difference $A \setminus B$ using addition and multiplication. (Note that this is not the difference $A - B$.

5) Draw a diagram which illustrates the associativity property $A + (B + C) = (A + B) \cdot A + C$ for addition.

6) Draw a diagram which illustrates the associativity property $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ for multiplication.

7) Draw a diagram which illustrates the distributive property $A \cdot (B + C) = A \cdot B + A \cdot C$.

8) Which sets have an inverse A^{-1} in the sense that $A \cdot A^{-1} = 1 = X$?

Paradoxa

1. Objective

We look at examples of paradoxical statements which are at the heart of the incompleteness theorems.

2. The Russell paradox

> The set of all sets which do not contain themselves as elements.

a) Verify that if X contains itself as an element then X contains itself as an element.

b) Verify that if X does not contain itself as an element then X does not contain itself as an element.

3. The barber paradox

> A barber in Cambridge only shaves people who do not shave themselves. Does the barber shave himself?

c) There is an elegant (non mathematical) solution to this paradox. Can you find it?

4. The liars paradox

The first version of the Liar Paradox seems have been found in the fourth century BC by Greek philosopher **Eubulides**, successor of Euclid. Somebody tells "I'm always lying". Does the person tell the truth? The paradox is attributed to the Cretan philosopher **Epimenides** who said "All Cretans are liars."

> The Cretean philopher Epimedes tells that all Cretans are liars.

Here is a variant of this paradox. Can you complete it?

> "The next sentence is .... The previous sentence is ...."
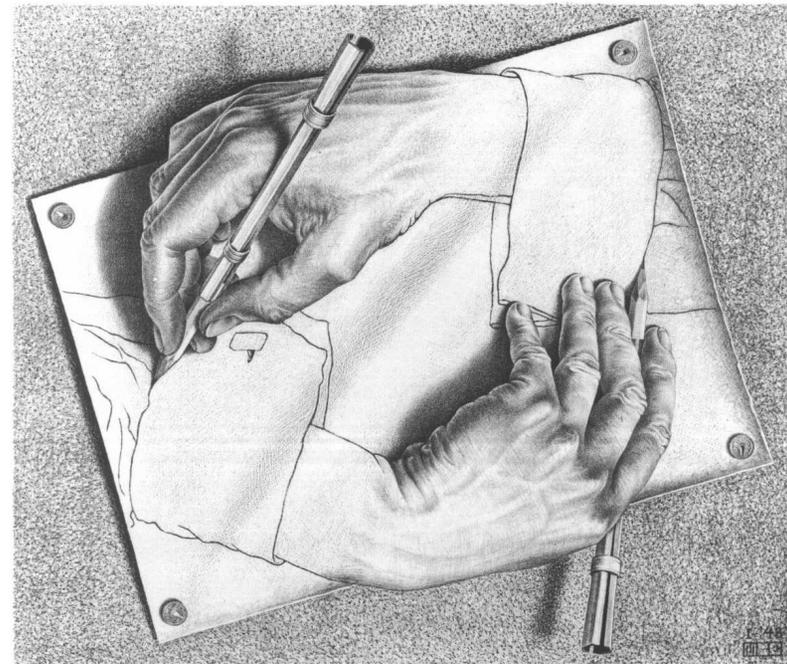
Cardinality

1. Objective

We explore some cardinality questions. We look at a continuous map from the unit interval onto the unit square. And explore whether it is a bijection.

1. Cardinality of fractions

Problem 1) Arrange the fractions in a suitable way so that you can count through them nicely. Can you get around the problem that certain fractions are counted several times?
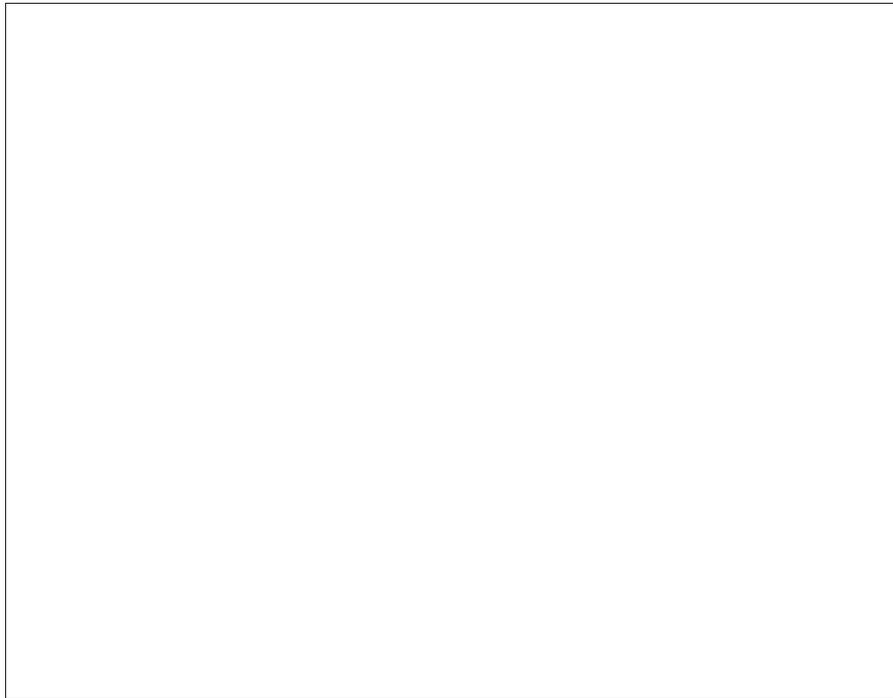
Problem 2) A number is called **algebraic** if it is a solution to a polynomial equation

$$p(x) = 0 .$$

where the polynomial has integer coefficients. Can you setup an argument that counts all the polynomials? This establishes that one can count all the algebraic numbers and therefore that there are numbers which are not algebraic.