Lecture 7: Set Theory and Logic

In this lecture, we focus on the work of the two mathematicians: **Georg Cantor** and **Kurt Gödel**. Both changed how to think about mathematics and the mathematical community needed time to interact. Hilbert said about Cantor "Nobody will drive us from the paradise that Cantor has created for us". Cantor made clear what "cardinality" is and showed us that different infinities exist and that some infinities are the same even so we might believe they are not.

Counting: Set theory

We first see that we can compute with sets like with numbers. We can add sets (the symmetric difference) and multiply sets (the intersection). With these two operations, we prove the familar rules

$$A + B = B + A, A \cdot B = B \cdot A, A \cdot (B + C) = A \cdot B + A \cdot C$$

This is called Boolean algebra. Which set plays the role of 0 and which set plays the role of 1 the multiplicative unit?

Counting: Hilbert's Hotel

Hilbert's hotel is located on route 8. It has countably many rooms numbered $1, 2, 3, \dots$. The hotel is fully booked. As a newcomer arrives. David, the hotel manager is mortified. especially because Daivd is a VIP wanting a room with a small number. David has an idea and moves guest in room i to room $i + 1$ and gives the newcomer the first room 1.

An other day, the hotel is empty but a large group arrives. They are the "fractions" on their way to a cardinal match with the "squares". Can David accomodate them? He thinks hard and finally manages.

In the summer, the "reals" appear. David is not there but has George, the apprentice is in the office. The group consists of all real numbers between 0 and 1. Can George acomodate them? As much as he tries to shift and renumber, he can not do it.

Counting: the interval

The interval $(-1, 1)$ has the same cardinality than the real line. The function $f(x) = \tan(\pi x/2)$ maps the interval $(-1, 1)$ onto the real line, one to one.

The square $(0, 1) \times (0, 1)$ has the same cardinality than the real line. A bijection can be constructed by $f(0.a_1a_2a_3a_4...) = (0.a_1a_3a_5, 0.a_2a_4a_6...)$.

Arithmetic with sets

One can calculate with sets as with numbers. They form a "Boolean ring".

Addition: $A + B = A \Delta B$ with the zero element $\emptyset$
Multiplication: $A \cdot B = A \cap B$ with the one element Ω.

All the rules of the real numbers apply but there are additional consequences which appear a bit strange $A + A = 0$ and $A^2 = A \cdot A = A$. This means that A is its own additive inverse.

Paradoxa

We have seen a few paradoxa like the **Liars paradox**, the **barbers paradox**, the surprise exam problem. Here is an other, the **Berry paradox** which comes somehow close to the Goedel numbering:

The smallest integer not definable in less than 11 words.

The problem is that this number is defined with 10 words. This looks like a stupid example but it illustrates that there are properties of numbers like "the shortest way to describe the number" which is not computable.

Lecture 7: Set Theory and Logic

Set theory studies sets, the fundamental building blocks of mathematics. While **logic** describes the language of all mathematics, set theory provides the framework for additional structures.

In **Cantorian set theory**, one can compute with subsets of a given set X like with numbers. There are two basic operations: the **addition** $A+B$ of two sets is defined as the set of all points which are in exactly one of the sets. The **multiplication** $A \cdot B$ of two sets contains all the points which are in both sets. With the symmetric difference as addition and the intersection as multiplication, the subsets of a given set X become a **ring**. One calls it a **Boolean ring**. It has the property $A+A=0$ and $A \cdot A = A$ for all sets. The zero element is the empty set $\emptyset = \{\}$. The additive inverse of A is the complement $-A$ of A in X. The multiplicative 1-element is the set X because $X \cdot A = A$. As in the ring of integers, the addition and multiplication on sets is commutative and multiplication does not have an inverse in general.

Two sets A, B have the **same cardinality**, if there exists a one-to-one map from A to B. For finite sets, this means that they have the same number of elements. Sets which do not have finitely many elements are called **infinite**. Do all sets with infinitely many elements have the same cardinality? The integers Z and the natural numbers N for example are infinite sets which have the same cardinality: $f(2n) = n, f(2n+1) = -n$ establishes a bijection between N and Z. Also the rational numbers Q have the same cardinality than N. Associate a fraction p/q with a point (p,q) in the plane. Now cut out the column $q=0$ and run the Ulam spiral on the modified plane. This provides a numbering of the rationals. Sets which can be counted are called of cardinality $\aleph_0$.

Does an interval have the same cardinality than the reals? Even so an interval like $(-\pi/2, \pi/2)$ has finite length, one can bijectively map it to the real lines with the tan map. Any two intervals of positive length, have the same cardinality. It was a great moment of mathematics, when **Georg Cantor** realized in 1874 that the interval $(0,1)$ does not have the same cardinality than the natural numbers. His argument is ingenious: assume, we could count the points $a_1, a_2, \dots$. If $0.a_{i1}a_{i2}a_{i3}...$ is the decimal expansion of a_i, define the real number $b = 0.b_1b_2b_3...$, where $b_i = a_{ii} + 1 \bmod 10$. Because this number b does not agree at the first decimal place with a_1, at the second place with a_2 and so on, the number b does not appear in that enumeration of all reals. It has positive distance at least 10^{-i} from the i'th number (and any representation of the number by a decimal expansion which is equivalent). This is a contradiction. The new cardinality, the **continuum** is also denoted $\aleph_1$. The reals are **uncountable**. This gives elegant proofs like the existence of **transcendental number**, numbers which are not algebraic, the root of any polynomial with integer coefficients: algebraic numbers can be counted.

Similarly as one could establish a bijection between the natural numbers N and the integers Z, there is a bijection f between the interval I and the unit square: if $x = 0.x_1x_2x_3...$ is the decimal expansion of x then $f(x) = (0.x_1x_3x_5..., 0.x_2x_4x_6...)$ is the bijection. Are there cardinalities above $\aleph_0$ and $\aleph_1$? Cantor answered also this question. He showed that for an infinite set, the set of all subsets has a larger cardinality than the set itself. How does one see this? Assume there is a bijection $x \to A(x)$ which maps each point to a set $A(x)$. Now look at the set $B = \{x \mid x \notin A(x)\}$ and let b be the point in X which corresponds to B. If $y \in B$, then $y \notin B(x)$. On the other hand, if $y \notin B$, then $y \in B$. The set B does appear in the "enumeration" $x \to A(x)$ of all sets. The set of all subsets of N has the same cardinality than the continuum: $A \to \sum_{j \in A} 1/2^j$ provides a map from $P(N)$ to $[0,1]$. The set of all **finite subsets** of N however can be counted. The set of all subsets of the real numbers has cardinality $\aleph_2$, etc.

Is there a cardinality between $\aleph_0$ and $\aleph_1$? In other words, is there a set which can not be counted and which is strictly smaller than the continuum in the sense that one can not find a bijection between it and R? This was the first of the 23 problems posed by Hilbert in 1900. The answer is surprising: one has a choice. One can accept either the "yes" or the "no" as a new axiom. In both cases, Mathematics is still fine. The nonexistence of a cardinality between $\aleph_0$ and $\aleph_1$ is called the **continuum hypothesis** and is usually abbreviated CH. It is independent of the other axioms making up mathematics. This was the work of **Kurt Gödel** in 1940 and **Paul Cohen** in 1963. For most mathematical questions, it does not matter whether one accepts CH or not. The story of exploring the consistency and completeness of axiom systems of all of mathematics is exciting. Euclid axiomatized all of Euclidean geometry, Hilbert's goal was much more ambitious, to find a set of axiom systems for all of mathematics. The challenge to prove Euclid's 5'th postulate is paralleled by the quest to prove the CH. But the later is much more fundamental and striking because it deals with **all of mathematics** and not only with a particular field of geometry. Here are the **Zermelo-Frenkel Axioms** (ZFC) including the Axiom of choice (C) as established by **Ernst Zermelo** in 1908 and **Adolf Fraenkel** and **Thoral Skolem** in 1922.

Extension	If two sets have the same elements, they are the same.
Image	Given a function and a set, then the image of the function is a set too.
Pairing	For any two sets, there exists a set which contains both sets.
Property	For any property, there exists a set for which each element has the property.
Union	Given a set of sets, there exists a set which is the union of these sets.
Power	Given a set, there exists the set of all subsets of this set.
Infinity	There exists an infinite set.
Regularity	Every nonempty set has an element which has no intersection with the set.
Choice	Any set of nonempty sets leads to a set which contains an element from each.

The **axiom of choice (C)** has a nonconstructive nature which can lead to seemingly paradoxical results like the **Banach Tarski paradox**: one can cut the unit ball into 5 pieces, rotate and translate the pieces to assemble two identical balls of the same size than the original ball. Gödel and Cohen showed that the axiom of choice is logically independent of the other axioms ZF. Other axioms in ZF have been shown to be independent, like the **axiom of infinity**. A **finitist** would refute this axiom and work without it. It is surprising what one can do with finite sets. The **axiom of regularity** excludes Russellian sets like the set X of all sets which do not contain themselves. The **Russell paradox** is: Does X contain X? It is popularized as the **Barber riddle**: a barber in a town only shaves the people who do not shave themselves. Does the barber shave himself?

A complete axiomatization of mathematics is never complete because of **Gödels theorems** of 1931. They deal with **mathematical theories**. They are assumed to be sufficiently strong meaning that one can do at least basic arithmetic in them and call it simply **a theory**:

First incompleteness theorem:
In any theory there are true statements which can not be proved within the theory.

Second incompleteness theorem:
In any theory, the consistency of the theory can not be proven within the theory.

The proof uses an encoding of mathematical sentences which allows to state liar paradoxical statement "this sentence can not be proved". While the later is an odd recreational entertainment gag, it is the core for a theorem which makes striking statements about mathematics. These theorems are not limitations of mathematics; they illustrate its infiniteness. How awful if one could build axiom system and enumerate mechanically all possible truths from it.