Lecture 6: Calculus

Calculus formalizes the process of **taking differences** and **taking sums**. Differences measure **change**, sums explore how things **accumulate**. The process of taking differences has a limit called **derivative**. The process of taking sums will lead to the **integral**. These two processes are related in an intimate way. In this lecture, we look at these two processes in the simplest possible setup, where functions are evaluated only on integers and where we do not take any limits.
About 25'000 years ago, numbers were represented by units like

$$1, 1, 1, 1, 1, 1, \dots$$

for example carved in the Ishango bone. It took thousands of years until numbers were represented with symbols like today

$$0, 1, 2, 3, 4, \dots .$$

Using the modern concept of function, we can say $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$ and mean that the **function** f assigns to an input like 1001 an output like $f(1001) = 1001$. Lets call $Df(n) = f(n+1) - f(n)$ the **difference** between two function values. We see that the function f satisfies $Df(n) = 1$ for all n. We can also formalize the summation process. If $g(n) = 1$ is the function which is constant 1, then $Sg(n) = g(0) + g(1) + \dots + g(n-1) = 1 + 1 + ... + 1 = n$. We see that $Df = g$ and $Sg = f$. Now lets start with $f(n) = n$ and apply **summation** on that function:

$$Sf(n) = f(0) + f(1) + f(2) + \cdots + f(n-1) .$$

In our case, we get the following values:

$$0, 1, 3, 6, 10, 15, 21, \dots .$$

The new function g satisfies $g(1) = 1, g(2) = 3, g(2) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n+1) - g(n) = f(n) .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$ which indeed is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. Can you do it?
When **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 numbers $1 + 2 + \cdots + 100$. Gauss found the answer immediately by pairing things up: to add up $1 + 2 + 3 + \dots + 100$ he would write this as $(1 + 100) + (2 + 99) + \cdots + (50 + 51)$ leading to 50 terms of 101 to get for $n = 101$ the value $g(n) = n(n-1)/2 = 5050$. Taking differences again is easier $Dg(n) = n(n+1)/2 - n(n-1)/2 = n = f(n)$.
Lets add up the new sequence again and compute $h = Sg$. We get the sequence

$$0, 1, 4, 10, 20, 35, ...$$

These numbers are called the **tetrahedral numbers** because one use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls for example to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(n) = n(n-1)(n-2)/6}$. We see (see worksheet) that summing the differences gives the function in the same way as differencing the sum:

$$\boxed{SDf(n) = f(n) - f(0), \quad DSf(n) = f(n) .}$$

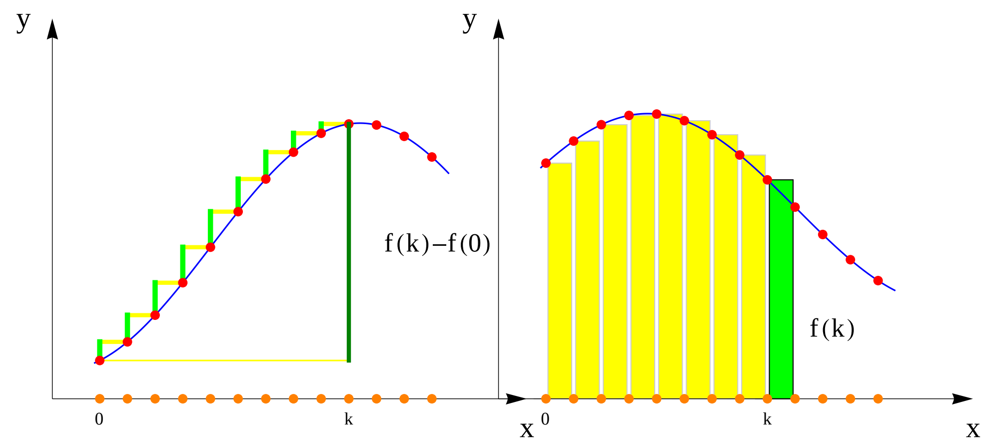
Proof.

$$SDf(kh) = h\sum_{k=0}^{n-1}[f((k+1)h) - f(kh)]/h = f(kh) - f(0) ,$$

$$DSf(kh) = [h\sum_{k=0}^{n-1} f((k+1)h) - h\sum_{k=0}^{n-1} f((k)h)]\frac{1}{h} = f(kh) .$$

This is an arithmetic version of the **fundamental theorem of calculus**. The process of adding up numbers will lead to the **integral** $\boxed{\int_0^x f(x)\,dx}$. The process of taking differences will lead to the **derivative** $\boxed{\frac{d}{dx} f(x)}$.

$$\boxed{\int_0^x \frac{d}{dt} f(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)}$$



Theorem: Sum the differences and get $SDf(kh) = f(kh) - f(0)$

Theorem: Difference the sum and get $DSf(kh) = f(kh)$

If we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

The calculus you have just seen, contains the essence of single variable calculus. This core idea will become more powerful and natural if we use it together with the concept of limit.

1 **Problem:** The Fibonnacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? We take the difference between successive numbers and get the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, ...$$

which is the same sequence again. We can deduce from this recursion that f has the property that $\boxed{Df(x) = f(x-1)}$.

2 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, ...$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, ...$ What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

We can study differences to understand sums.

The next problem illustrates this too:

3 **Problem:** Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0

.

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

4 **Problem:** The function $f(n) = 2^n$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \dots$. It leads to the sequence of numbers

n=	0	1	2	3	4	5	6	7	8	...
f(n)=	1	2	4	8	16	32	64	128	256	...

We can verify that f satisfies the equation $Df(x) = f(x)$. because $Df(x) = 2^{x+1} - 2^x = (2-1)2^x = 2^x$.
This is an important special case of the fact that

The derivative of the exponential function is the exponential function itself.

The function 2^x is a special case of the exponential function when the Planck constant is equal to 1. We will see that the relation will hold for any $h > 0$ and also in the limit $h \to 0$, where it becomes the classical exponential function e^x which plays an important role in science.