Games for Thinkers

WFF 'N PROOF®

ON-SETS®

The Game of Set Theory

by

Layman E. Allen, Peter Kugel, Martin F. Owens

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TABLE OF CONTENTS

EQUIPMENT

16 color cards
2 playing mats
8 color cubes (each imprinted with blue, red, green, and orange dots)
4 red-symbol cubes (each imprinted with the symbols, $\cup$, $\cap$, $-$, and $'$)
3 blue-symbol cubes (each imprinted with $\underline{\vee}$, $\underline{\wedge}$, $\subseteq$, and $=$)
3 numeral cubes (each imprinted with 1, 2, 3, 4, and 5)
1 instruction manual
1 timer

Cubes and timer made in West Germany.

PREFACE

The first version of ON-SETS was designed in 1965–66 by the writer, then a faculty member at Yale Law School, in collaboration with Peter Kugel and Martin F. Owens, then on the staff of Technical Operations Incorporated, Burlington, Mass., and the original ON-SETS kit was first published in 1966. Since then it has been played in hundreds of schools and thousands of homes throughout the United States and the rest of the world, and as should be expected, there have been many useful suggestions for improving the game during the three years of experience with it.

One of the unanticipated benefits of moving to the University of Michigan has been the discovery of a now-treasured colleague to whom there is a special indebtedness in the preparation of this revised manual for ON-SETS—Joan Ross, Research Associate at the Mental Health Research Institute, University of Michigan. For her efforts in coaching the team from Ann Arbor, Mich., that won the Junior High School Championship at the 1969 National Academic Games Olympics, Mrs. Ross was nicknamed "Mother Superior" by the imaginative Michigan contingent. She is also a "Colleague Suprema" as many discernible traces spread throughout this manual will confirm.

Although the ON-SETS games have profited from the contributions of many persons, I alone should be saddled with responsibility for any imperfections in what appears here. It is too much to hope that this entire manual is free of error. I shall be grateful to any readers who are kind enough to send to me at the Mental Health Research Institute, University of Michigan, Ann Arbor, Mich. 48104, any suggestions for corrections that need to be made or any suggestions for improving the next version of the ON-SETS games.

Layman E. Allen*
Ann Arbor, Mich.
July, 1969

*Associate Professar af Law, Law Schaal; Research Social Scientist, Mental Health Research Institute, University of Michigan.

1. Introduction

ON-SETS is a game that teaches some of the basic ideas of set theory. Set theory is a branch of mathematics that underlies the other branches of mathematics. Hence understanding set-theoretic ideas is necessary for understanding much of modern mathematics. Because of this, many math courses and books begin with a short introduction to set theory, but this introduction is seldom sufficient for an understanding of the rest of the course or book if one has not had some previous acquaintance with set theory. The use of the set theoretic approach to modern mathematics is not mere window dressing; it makes modern mathematics both simpler and more powerful.

To learn a branch of mathematics, it is not enough merely to learn a series of facts. One must also learn how to use these facts. Mathematics is like a language (it is the language of science) and learning a branch of mathematics is, in at least one respect, like learning French. In learning French, it is not enough to know the grammar and to memorize the dictionary. One must learn to use the facts that are described in the grammar and dictionary. The same is true in learning mathematics; and in particular, set theory.

All the ideas of set theory that are used in ON-SETS are explained in this manual. The game provides a medium for practicing the use of these ideas. Although one might want to play ON-SETS in order to master the ideas of set theory, we hope that the main reason people will play it is because it is fun, and that it remains fun long after the subject has been mastered.

This manual contains various sections that are intended to serve various purposes for different people who want to learn to play ON-SETS. You should read only those parts that suit your needs.

Section 2 contains a brief description of the Basic ON-SETS game, while section 3 sets forth the rules for playing both Basic ON-SETS and Advanced ON-SETS. In addition, section 3 is intended for reference after the game has been learned. Somebody who already knows the subject of set theory may find enough in this section and section 2 to learn the game. A person who knows nothing about either the game or the subject of set theory may want to read these sections through quickly before reading the other sections in order to get a general idea of how the game is played.

Sections 2 and 3 in combination with 4, 5, 6, and 7 describe both the form of the game and enough about set theory to enable a person to play either Basic ON-SETS or Advanced ON-SETS. The game-rule ideas and the set-theoretic ideas are illustrated in the context of play by examples of challenges, puzzles, and sample games.

By no means should you feel that this manual has to be read from cover to cover. Skip to the part that makes sense to you. If you find sections 3–7 too complicated to start with right away, you may wish to try some of the simpler games described in section 8. These games are much simpler than Basic ON-SETS, but they still give practice in handling some of the ideas of set theory.

You may find it helpful to work through the exercises that are sprinkled throughout the manual (most readers probably will). On the other hand, some readers will undoubtedly not need to do so.

If you wonder what set theory is all about, you might start out with section 9 which contains a brief summary of that part of set theory that is involved in playing Basic ON-SETS and Advanced ON-SETS.

Finally, if you wonder what the point of learning set theory at all might be, you should turn first to section 10 where you will find a brief answer to this question.

If you are a teacher and want to consider the classroom use of ON-SETS, section 8 describes a sequence of games, each introducing some new material or providing practice with familiar material in a new form. Some of these games are simple enough to play in kindergarten with little preparation. But even an adult, with no classroom commitment at all, may find it easier to learn both the game and the subject with the sequence of simpler games described in section 8.

2. Description of Basic ON-SETS Game

2.1 Number of Players: two or more

ON-SETS may be played by as few as two players. On the other hand, it may be played by as many players as can comfortably gather around the table to see the cards and cubes. A three-player game is recommended for classroom use.

2.2 Levels of Play

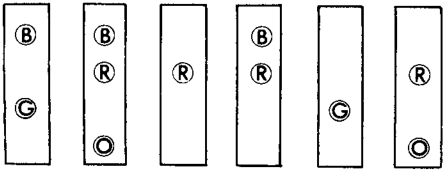
ON-SETS may be played at many levels of difficulty. The level of game is determined by the cubes used, the number of cards in the Universe, the rules, and how well the players understand the ideas involved in the game. Basic ON-SETS is played with three red-symbol cubes, three green numeral cubes and four multi-color cubes. The blue-symbol cubes are not used. The Universe (of cards) must contain at least six cards. In general a larger Universe will give rise to a more complex game.

There is also a programmed series of 28 simpler games which introduce various features of ON-SETS. They can be used for learning parts of the game rules as well as some set-theoretic ideas used in the game. These simpler games are discussed in section 8. Some of these introductory games are simple enough to be used in kindergarten. The Basic Game, however, frequently requires careful thought, and, there is an Advanced ON-SETS game that is challenging even for intelligent adults. In Advanced ON-SETS, all the cubes are used (including the blue-symbol cubes). The rules of Advanced ON-SETS are exactly the same as the rules for Basic ON-SETS except that Rule 3.4′ is substituted for Rule 3.4.

2.3 Explanation of the Symbols on the Cubes

Before the cubes are shaken in the beginning of an ON-SETS game, the player to the right of the first player shuffles the deck of cards and deals out (face-up) at least six cards. The cards are the Universe for that play of the game. The sets named in the play are always sets of cards from that Universe.

The color cubes are the names of sets of cards. A cube showing a red dot is the name of the set of cards (in the Universe) that have red dots. A cube showing a green dot is the name of the set of cards bearing green dots. For example if the Universe looks like this:



Card 1 Card 2 Card 3 Card 4 Card 5 Card 6

a color cube with green showing names the set consisting of exactly two cards: card 1 and card 5.

The red-symbol cubes display symbols for operations on sets. The $\boxed{\cup}$, which stands for the operation of set-union, is called "cup". The $\boxed{\cap}$, which stands for the operation of set-intersection, is called "cap". The $\boxed{-}$, which stands for the operation of set-difference, is called "minus". These three represent binary operations on sets. That means that when they are placed between the names of two sets, the resulting expression is the name of a set. The other symbol on the red-symbol cubes, the $\boxed{'}$, stands for the operation of complementation; it is called "prime". It is not a binary operation but a unary operation. When it is placed *after* the name of a set, the result is also the name of a set.

The blue-symbol cubes display both Set-Names and symbols for relations between sets. The $\boxed{\text{V}}$ is a Set-Name and stands for the Universe —the set of cards turned face up on the table. The $\boxed{\Lambda}$ is a name of the empty set—the set which has no elements. The $\boxed{=}$ represents the relation of set-identity between sets; and the $\boxed{\subseteq}$, the relation of set-inclusion. When '$=$' (or '$\subseteq$') is placed between the names of two sets, the result is not a set but a statement about sets. The $\boxed{=}$ and the $\boxed{\subseteq}$ can *not* be used in building Set-Names.

The meanings of these operations and relations are explained in section 4.

2.4 Aim in Playing: to win by correctly challenging or by being incorrectly challenged

In ON-SETS you can win either (a) by correctly challenging a Flub that another player has made or (b) by being incorrectly challenged by another player. An important feature of ON-SETS is that you can win by challenging at any time (if somebody has Flubbed). You do not have to wait until it is your turn to play to make a challenge. Consequently, because in a three-player game two players are competing to challenge first, the play is more exciting than in a two-player game, where such competition is absent.

2.5 Start of Play

A player is chosen to be the first player or Shaker. The player to his right builds a Universe. He does so by shuffling the deck of cards and turning at least six cards face up on the table. The Shaker then rolls out a shaking set of cubes consisting of the three green numeral cubes, three of the red-symbol cubes, and four of the color cubes. The resulting symbols facing upward on the cubes are the Resources for that play of the game. In Basic ON-SETS, no blue-symbol cubes are used.

In Advanced ON-SETS, all cubes are used. The Shaker must decide whether to set a Goal or say "No Goal". If he does set o Goal, he places in the Goal section of the playing mat numeral cubes as the Goal and places any unused numeral cubes in the Forbidden section of the playing mat.

2.6 Play

After o Goal has been set, the players take turns moving cubes from Resources to the Forbidden, Permitted or Required sections of the playing mat.

2.7 End of Play

Play ends when there is a challenge, when a player asserts that a Solution can be built with one more cube from Resources, or when a player says "No Goal".

2.8 Summary of Play

When it is the first player's turn to play, he does one of the following:

(a) declares "No Goal", or

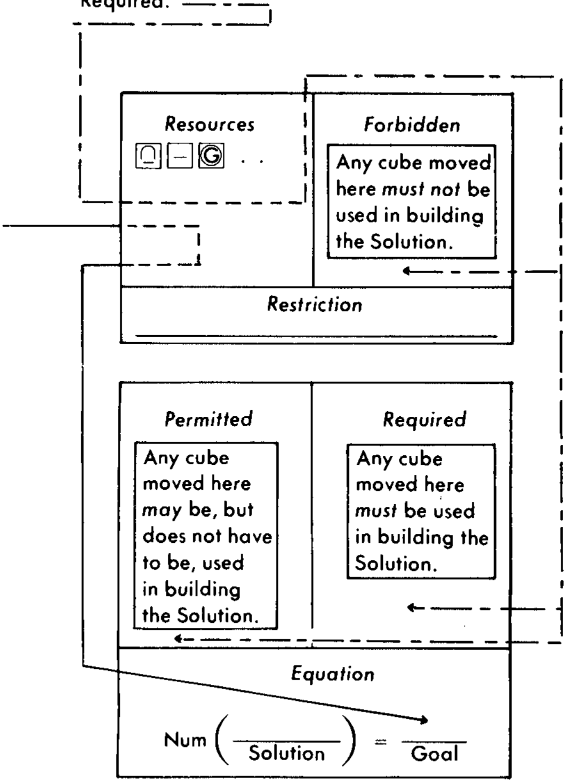
(b) sets a Goal.

After the first play when it is a player's turn, he does one of the following:

(a) declares that a Solution is possible with just one more cube from Resources,

(b) challenges the previous move as being a Flub, or

(c) moves a cube from the Resources to either Forbidden, Permitted, or Required.



3. Rules of ON-SETS[1]

3.1 Goal Rule

On your shake, you must either set a Goal in the Goal section of the playing mat or say "No Goal". The Goal must consist only of numbers. The sum of two numbers is indicated by placing the cubes in a horizontal line. The product of two numbers is indicated by placing the cubes in a vertical line. The negative of a number is indicated by placing the cube so that its numeral is upside down. After setting a Goal, all left-over numeral cubes are put into the Forbidden section.

Comments

(1) The Goal is the right side of an Equation. When it is possible to set a Goal for which a Solution (see 3.4 below) can be built from the remaining Resources, then in order to avoid Flubbing you must set such a Goal and you must not say "No Goal".

---

[1] Two other games in the WFF 'N PROOF series have essentially the same game rules as ON-SETS. They are EQUATIONS (about mathematics) and the PROOF games of WFF 'N PROOF (about mathematical logic). Because of this similarity, once you have learned to play ON-SETS you will also be well down the road toward learning these other two games.

(2) Examples of Goals:

[2][3] means $2 + 3$ [or 5].

[2]
[3] means 2×3 [or 6].

[2][ᘔ] means $2 + {}^{-}2$ [or 0].

[2]
[1][3] means $2 \times (1 + 3)$ [or 8].

(3) The Goal is not changed after it has been set.

3.2 Move Rule

After the Goal has been set, play progresses in a clockwise direction. When it is your turn to play, you must either challenge, assert (without challenging) that a Solution can be built with one more cube from Resources, or make one of the following moves:

(A) move a cube from Resources to the Forbidden section, or

(B) move a cube from Resources to the Permitted section, or

(C) move a cube from Resources to the Required section.

When it is your turn you are not permitted to pass.

Comment

By their moves, the players shape the Solution. Cubes are never moved after they have been moved from the Resources area.

3.3 Bonus Rule

On your turn to play you may take a bonus move before making a regular move (before setting a Goal or moving a cube to the Forbidden, Permitted, or Required sections). A bonus move consists of saying "Bonus" and moving one cube from Resources to the Forbidden section. If you do not say "Bonus" before moving the cube to the Forbidden section, the move does not count as a bonus move but as a regular move to Forbidden.

Comment

The Bonus Rule has the effect of allowing a Mover to move two cubes from the Resources, but one of them *must* go into the Forbidden section. Both of them *may* go into Forbidden.

3.4 Solution Rule

The Solution is the name of a set. The number of cards in that set must be equal to the Goal. In attempting to build a Solution

(A) you must not use any of the cubes in the Forbidden section;

(B) you may use as many of the cubes in the Permitted section as you like;

(C) you must use all of the cubes in the Required section;

(D) you may always use at least one cube from Resources: you may use at most one cube from Resources when there has been an A-claim challenge or a C-claim challenge that stems from a previous A-claim violation (see the Flubbing Rule in section 3.5 below), and you may use as many Resources as you like when there has been a P-claim challenge, a C-claim challenge that stems from a previous P-claim violation, or a No-Goal challenge (note that on a No-Goal challenge you must set a Goal as well as build a Solution); and

(E) you may insert parentheses wherever you want to put them (to show in which order the operations are to be performed).

Comment

Since several Resource cubes may show the same symbol, it is possible to have a $\boxed{\underline{\cup}}$ in Forbidden which *must not* be used in the Solution at the same time that there is a $\boxed{\underline{\cup}}$ in Required which *must* be used.

3.5 Flubbing Rule

If you do either of the following, you have Flubbed:

(A) you declare "No Goal" when, in fact, you could have set a Goal for which there was a Solution;

(B) by your move you violate one of the following C-A-P claims:

C On this turn I Cannot correctly challenge the previous move.

A If possible, I am Avoiding by this move allowing a Solution to be built with at most one more cube from Resources.

P It is still Possible for the remaining Resources to be so played that a Solution can be built.

Comments

(1) The P-claim means that you Flub if you make a move that destroys all possibilities for building a Solution.

(2) The A-claim means that you Flub if you make a move that permits a Solution to be built with just one more cube from Resources when you could have made a move that both avoided doing so and at the same time fulfilled the P-claim. Of course, when only two cubes are left in Resources, you may have to move one of them into the Permitted or the Required section and permit a Solution to be built with just one more cube from the Resources, because Forbidding either cube violates the P-claim. This is not an A-claim violation because in such circumstances it is not possible to avoid allowing a Solution to be built with just one more cube from the Resources without violating the P-claim.

(3) The C-claim means that once a Flub is made, every subsequent move is a Flub because every subsequent Mover could have correctly challenged. Since only the most recent Flub may be challenged, the C-claim makes it possible to win by laying a trap: make a deliberate Flub, and as soon as the next player moves, challenge him for failing to challenge you.

3.6 Challenge Rule

Whether or not it is your turn, you may at any time challenge the other player who has just completed a move or has just said "No Goal". You do so by saying "Challenge" and specifying which kind of Flub you think the Mover has made. The move of set-

ting a Goal is completed when the Mover touches a mat with the third numeral cube. The move of a cube to Forbidden, Permitted, or Required is completed when the Mover touches a mat with the cube. Prior Flubs are insulated by later ones; therefore you cannot challenge any player except the one who has just completed his play.

Comments

(1) A challenge cannot be retracted once a player has said "Challenge".

(2) To determine priority in those rare cases where two players say "Challenge" simultaneously, a coin should be placed in the center of the table when the cubes are first rolled. The first of the simultaneously-challenging players to pick up the coin shall be the Challenger; the other player shall be the Third Party.

3.7 Burden-of-Proof Rule

After a challenge, the burden of proof is cast upon the player who, in the particular situation, is claiming that a Solution can be built. The burden of proof is sustained by writing a Solution on a sheet of paper.

Comments

(1) A Solution must, of course, satisfy the conditions imposed by the Solution rule and the

previous plays of the cubes into the Forbidden, Permitted, or Required sections.

(2) Sometimes the burden of proof will be upon the Challenger—namely, when the Challenger alleges that there has been a Flub by virtue of a false No-Goal declaration, an A-claim violation, or a C-claim violation that stems from a previous A-claim violation. On the other hand, sometimes the burden will be upon the Mover—namely, when the Challenger alleges that there has been a Flub by virtue of a P-claim violation or a C-claim violation that stems from a previous P-claim violation.

(3) When a Challenger has alleged an A-claim violation or a C-claim violation that stems from a previous A-claim violation, he also has the burden of proving that there was an alternative move that (a) did not allow a Solution to be built with at most one more cube from the Resources, and (b) did not violate the P-claim.

(4) If there has been a challenge of a No-Goal declaration, the burden of proof is upon the challenger, who, from the Resources, must not only build a Solution but also set a Goal.

3.8 Correctness Rule

After a challenge, a player is Correct if and only if

(A) he has the burden of proving the existence of a Solution and he sustains it (by writing one), or

(B) he does not have the burden of proving the existence of a Solution (somebody else has the burden), and nobody sustains that burden of proof.

3.9 Challenge-Scoring Rule

If there has been a challenge, then

(A) the Third Party (T) must join either the Challenger (C) or the Mover (M), and

(B) if the player that T joins has the burden of proving the existence of a Solution, then T must sustain the same burden of proof by independently writing a Solution, and

(C) if T is Correct, then T scores 1 point if he has joined C and 2 if he has joined M, and

(D) C scores 2 if C is Correct, and

(E) M scores 2 if M is Correct, and

(F) if anyone is Incorrect, then he scores 0.

Comments

(1) If the game involves four or more players, then all of the players other than the Mover and the Challenger are Third Parties.

(2) The effect of this scoring rule is usually (although not always) that one of the two players involved in a challenge scores 2 and the other 0. In some circumstances they both may wind up with 0.

(3) T can score 2 when he joins M. However, T can score at most 1 by joining C. This places a premium upon being the first player to challenge another's Flub.

3.10 Non-Challenge-Scoring Rule

If there has not been a challenge, then

(A) if a player has asserted that a Solution can be built with one more cube from Resources, but that there is no Flub, then

- **(1) each player who writes a Solution within the specified time limit (usually from one to two minutes) scores 1, and**
- **(2) if the player who has asserted that a Solution can be built cannot build one, he scores $^{-}1$, and**
- **(3) all other players score 0, and**

(B) if the first player has said that No Goal can be set, then each player scores 1.

Comment

The situation described in (A) will generally arise when (1) there is only one cube left in Resources or (2) the only Goals for which a Solution is possible can be satisfied by a single-cube Set-Name. When deciding whether to challenge in this situation, players should remember that the Goal-Setter can place a cube in Forbidden as a bonus play before setting the Goal.

3.11 Stalling Rule

At any time any other player can call "stall" on the player who is

(A) deciding whether to set a Goal or to declare "No Goal", or

(B) deciding whether to move a cube, to challenge, or to assert that a Solution can be built with one more cube from the Resources, or

(C) deciding whom to join after a challenge, or

(D) trying to build a Solution.

The stalling player then has some specified time (usually one to two minutes) to complete what he is doing. If he fails to meet the deadline, he loses one point, and another limited time period begins. If he fails to meet the second deadline, he loses another point; and so on.

Advanced ON-SETS

All the rules are the same except that 3.4′ replaces 3.4 and the shaking set contains all the cubes, including those imprinted with blue symbols.

3.4′ Solution Rule for Advanced ON-SETS

The Solution is either the name of a set or the combination of both the name of a set and a set of one or more statements about sets.

The first type of Solution is called a Simple Set-Name Solution (S-Solution), and the second type is called a Combination Set-Name and Restriction-Statement Solution (C-Solution).

(A) If one or more $\boxed{=}$ cubes or $\boxed{\subseteq}$ cubes are played in Required, then you must build a C-Solution that contains a Restriction Statement for every $\boxed{=}$ cube and for every $\boxed{\subseteq}$ cube played in Required.

(B) If no $\boxed{=}$ cubes or $\boxed{\subseteq}$ cubes are played in Required but some are played in Permitted, then you may build either an S-Solution or a C-Solution.

(C) If no $\boxed{=}$ cubes or $\boxed{\subseteq}$ cubes are played in either Permitted or Required and none are left in Resources, then you must build an S-Solution.

(D) If you attempt to build an S-Solution, then it must satisfy all the requirements of Rule 3.4 of the Basic ON-SETS game.

(E) If you attempt to build a C-Solution, then
   (1) you must use every cube played in the Required section in building the set of Restriction Statements, and
   (2) you must use every cube played in the Required section except the $\boxed{=}$ cubes and the $\boxed{\subseteq}$ cubes in building the Set-Name, and
   (3) of the cubes played in the Permitted section,
      (a) you may use as many as you like in building the Set-Name and
      (b) you may use as many as you like in building the set of Restriction Statements, and

(4) you must not use any of the cubes played in the Forbidden section either in building the Set-Name or in building the set of Restriction Statements, and

(5) you may always use one cube from Resources, but

(a) if there has been an A-claim challenge, a C-claim challenge that stems from a previous A-claim violation, or an assertion (without challenging) that a Solution can be built with one more cube from Resources, then you may use only one more cube from Resources and it may be used in building either the Set-Name, the set of Restriction Statements, or both, and

(b) if there has been a P-claim challenge, a C-claim challenge that stems from a previous P-claim violation, or a No-Goal challenge, then you may use as many Resources as you like in either the Set-Name, the set of Restriction Statements, or both, and

(6) you may insert parentheses wherever you want to put them, and

(7) every card that makes a Restriction Statement untrue must be removed from the Universe,

(8) within the restricted Universe that remains, the number of cards in the set named by the Set-Name must be equal to the Goal.

4. Ideas of ON-SETS

4.1 Set

A set may be named by a row of one or more cubes such as:

[R]

or

[R] [∪] [G] [∪] [B].

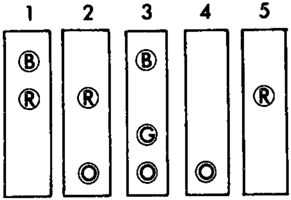
In general, a Set-Name tells us what properties a card must have to be included in a given set or collection. How *many* cards are actually in a set will depend on which cards have been turned up in the Universe for the game. Thus, the same Set-Name can be the name of different sized sets in different games.

The simplest kind of Set-Name consists of one cube only. A one-cube Set-Name can be made only with the color cubes.[1] The symbols on the red-symbol cubes ([∪] [∩] [−] [′]) can be used only in longer Set-Names.

The single-cube Set-Name is the name of the collection of all the cards that have the color showing (face-up) on the cube. Thus, for example, if we had the following sample Universe of cards on the table:

[1]This is the case in Basic ON-SETS. In Advanced ON-SETS, however, one-cube Set-Names can be made from [V] and [Λ], also.

Sample Universe



the Set-Name consisting of the red cube only, Ⓡ, would be the name of the collection that contained three cards, namely the first two and the last. The orange cube would be the name of a different collection, namely the collection of all the orange cards. This collection contains the middle three cards. Note that although both collections are the same size (they have the same number of cards) they do not contain the same cards. The blue cube would be the name of a collection that contains only two cards (Which two?)

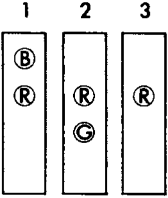
In playing ON-SETS, a Solution is a Set-Name that names a collection that contains exactly the number of cards stated in the Goal.

All other Set-Names in Basic ON-SETS are built by combining the color cubes with the red-symbol cubes. The way a one-cube expression names a set of cards is similar to the way that the name "George" called in a room, may name a set of people. If we call "George" we may get more than one person responding. In order to be more specific we might call "George Jones" or if we wanted more replies we might call "George or Mary". In both cases we have added something to the word "George" to call on what might be a different collection of people. Similarly, when we use more cubes we may form a more specific or a less specific Set-Name from a set of cubes.[1]

Exercises

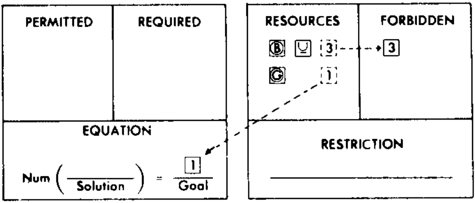
The situations that follow are simplified for purposes of illustration by having less than six cards in the Universe. In playing ON-SETS, however, the dealer will always turn up at least six cards.

Universe



[1] If only one player has the burden of proof, it can be sustained by physically building the Solution from the cubes. We shall continue in this section to show Solutions as made from the actual cubes. However if more than one player has the burden of proof (See Burden of Proof and Challenge Scoring rules, 3.7 and 3.9.), each player who has it must independently write a Solution.

Playing Mats



The player to the right of the first player has turned up the color cards. The Shaker has tossed the shaking set of cubes with the result shown above. He has set a Goal of 1. There are at least two ways of fishing out one color card. The first way is by constructing a Solution consisting of the green cube alone, which would draw out Card 2.

A. What is another Solution?

B. Which card will it fish out?

With this Universe and with the Resources given, if the Goal is 1, there are two Solutions.

A. The blue cube alone.

B. Card 1.

4.2 Operations on Sets (the red-symbol cubes)

4.2-A Union of Sets

The connective ∪ is called the "cup".* It can be placed between any two Set-Names to form a single new Set-Name. This new Set-Name names what is called the "union" of the two sets. The cup is a symbol that is used in very much the same way that the addition sign (+) is used in arithmetic. We place the addition sign between the names of two numbers (as in 4 + 5, or in (3 − 1) + (5 − 2)) to form the name of a single new number. Similarly, we place the cup between the names of two sets to form the name of a new set. The result of using the addition sign is called the "sum" of two numbers. The result of using the cup is called the "union" of two sets.

For example, we already know that a single color cube is a Set-Name so that we can use a cup between two such Set-Names, say Ⓡ and Ⓑ to form a single Set-Name:

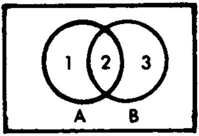
Ⓡ ∪ Ⓑ

This Set-Name is the name of a collection of cards, and this collection or set contains all the cards that are red or that are blue (as well as all that are

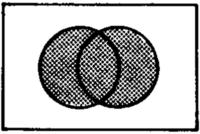
*The lines underneath cup and other symbols on the cubes do not appear in mathematics texts. They are used here to indicate which direction the cubes should be turned when used in forming a Set-Name.

both). It will usually be larger than the set that is denoted by either [R] or [B] alone. Notice that although there are three red cards and two blue cards in the sample Universe on page 26, there are only four cards in the union (even though 2 + 3 = 5). The reason for this, of course, is that there is one card that is both red and blue. As an exercise, determine how many cards the expression [G] [∪] [O] names in the sample Universe.)

We can represent sets by the space enclosed by circles



The rectangle represents the Universe. The space in each circle represents each set, and the overlapping part represents the set that contains the things that are in both sets. Set A contains the things in area 1 and the things in area 2. Set B contains the things in area 2 and the things in area 3. The union of Set A and Set B (that is, $A \cup B$) contains the things in area 1, the things in area 2, and the things in area 3. Alternatively, we can represent the union of two sets (represented by circles) by the area shaded in the following diagram:



We will use this kind of diagram (called a Venn diagram) to indicate what the other symbols ([∩], [−], and [′]) mean.

Note that more than one cup can be used in constructing an expression. Thus,

[G] [∪] [B] [∪] [R]

is an expression that is constructed by placing a cup between the two expressions:

[G] [∪] [B]

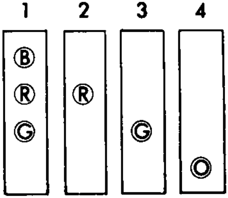
and

[R]

The written version of this Solution is '(G∪B) ∪R'. The cards that are in this set are all the cards that are in the set named by the first Set-Name *and* all the cards that are in the set named by the second Set-Name (Which ones are they in the sample Universe?)

Exercises

P1 (Problem 1)—Universe



The expression Ⓡ ∪ Ⓑ will fish out all the cards that have either red or blue or both on them (Cards 1 and 2 above).

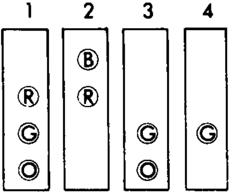
A. What is a Set-Name that will fish out just cards 1 and 4 above?

B. What is a Set-Name that will fish out just cards 1 and 3 from the set above?

P1-A. Ⓑ ∪ Ⓞ.

P1-B. Ⓖ, Ⓑ ∪ Ⓖ. (You have probably already discovered that to many of the questions asked there will be more than one appropriate response. This is such a question. There will frequently be other appropriate responses that are not listed.)

P2 (Problem 2)—Universse



Assume that all four colors are in the Resources on this roll and also a [∪].

A. Construct a Solution that will fish out all four cards.

B. Construct a Solution that will fish out two cards only.

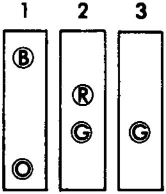
C. Is it possible to construct a Solution that will draw out only one of the above cards?

P2-A. [B] [∪] [G], [R] [∪] [G].

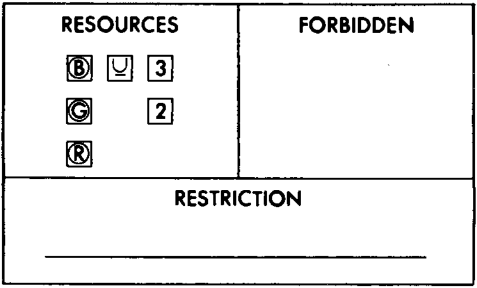
P2-B. [O], [R], [R] [∪] [B].

P2-C. Yes. [B].

P3 (Problem 3)—Universe



Playing Mat

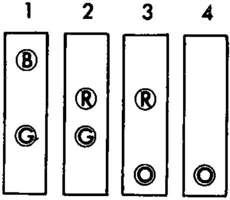


You are the Shaker in the situation shown above. You must now set a Goal. There are at least two possibilities. A Goal of 3 would be satisfied by fishing out all three cards with the Solution B ∪ G.

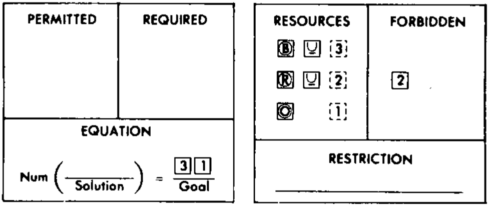
A. If you were to set a Goal of 2, what expressions would satisfy that Goal as Solutions?

P3-A. G, B ∪ R, R ∪ G.

P4 (Problem 4)—Universe



Playing Mats



Situation as shown above. You as the Shaker are now required to set a Goal. You have set a Goal of 4 by placing the [3] and the [1] on the Goal line (3 + 1 = 4). One Solution that would fish out all four cards is [B] [∪] [R] [∪] [O].

A. What Goals of smaller numbers could you have set and what expressions would satisfy them as Solutions? List two.

P4-A. Num ([B] [∪] [R]) = [3], Num ([R]) = [2].

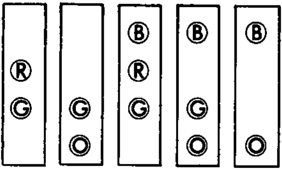
Remember that the GOAL

Num ________ = ______

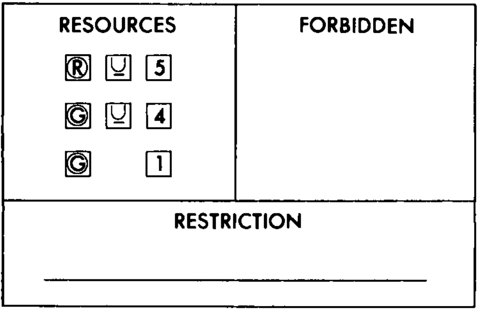
represents the *NUMBER*
of cards that are fished out by the *SOLUTION* constructed.

P5 (Problem 5)—Universe

1 2 3 4 5



Playing Mat

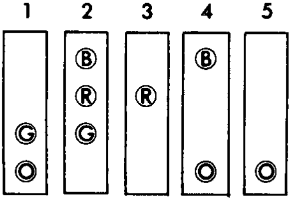


You are faced with the situation shown above and must set a Goal. After examining the cards and cubes it becomes apparent to you that all of the multi-cube expressions possible fish out the same number of cards.

A. So you set a Goal of ______ .

B. One expression that will name that many cards is [G] [∪] [G]. What is another?

P6 (Problem 6)—Universe



RESOURCES	FORBIDDEN
[B] [∪] [1] [R] [∪] [3] [O] [2]	
RESTRICTION ______________	

P5-A. 4.

P5-B. [R] [∪] [G] [∪] [G] or [R] [∪] [G].

List one multi-cube Solution that will fish out the appropriate number of cards to satisfy each of the following Goals for the situation shown above.

Goal		*Solution*
A.	5	
B.	4	
C.	3	

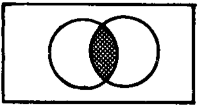
4.2-B Intersection of Sets

The symbol [∩] is called the "cap". It is used like the cup in that it is placed between two Set-Names to make a single new Set-Name as in:

[B] [∩] [R]

However, the meaning of the resulting Set-Name is quite different from the Set-Name formed with a cup. The Set-Name formed by placing the cap between two Set-Names is called the "intersection" of the two other sets. The intersection of two sets contains only the things that are in *both* of the sets. Thus in the sample Universe (see page 26), the above Set-Name names a set that contains only a single card, namely the card that is *both* blue and red. We can represent the intersection of two sets by the shaded portion of the following diagram:

P6-A. [R] [∪] [O].
P6-B. [B] [∪] [O].
P6-C. [B] [∪] [R].



The part that is common to both sets is the intersection. The idea of an intersection is very much like the idea of an intersection of two streets. The intersection is that part of the pavement that belongs to *both* streets.

One way of remembering the difference between cap and cup is to think of a cap as an upside down cup that holds as little as possible, while the cup held right side up contains as much as possible.

Notice that the order in which these connectives are used now makes a difference. Thus, there is a difference between what the Set-Name:[1]

[R] [∪] [G] [∩] [O]

means (where [∩] is placed between the Set-Names [R] [∪] [G] and [O]) and what the Set-Name:[2]

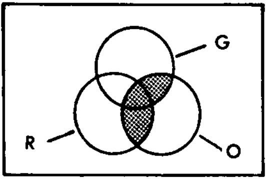
[R] [∪] [G] [∩] [O]

means (where [∪] is placed between the Set-Names [R] and [G] [∩] [O]).

[1] This would be written $(R \cup G) \cap O$.
[2] This would be written $R \cup (G \cap O)$.

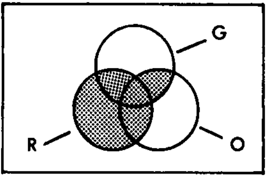
What the first Set-Name names can be illustrated like this:

R ∪ G   ∩   O



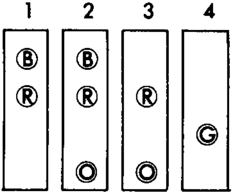
while what the second Set-Name names can be illustrated like this:

R   ∪   G ∩ O



Exercises

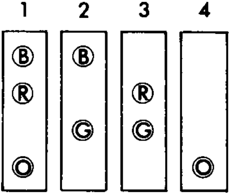
P1 (Problem 1)—Universe



The Set-Name [R] [∩] [B] will fish out cards 1 and 2 of the above because each of them is both red and blue.

A. Which cards will the Set-Name [R] [∩] [O] fish out?

P2 (Problem 2)—Universe



P1-A. 2 and 3.

PERMITTED	REQUIRED

EQUATION
Num (______ Solution) = 1 Goal

RESOURCES	FORBIDDEN
B ∪ 2	2
G ∩ 2	2
O 1	

RESTRICTION
______

As the Shaker you have set a Goal of 1 in the situation shown above. There are many Solutions that will fish out just one card. One such Solution is B ∩ O which will fish out only card 1 since it is the only card that is both blue and orange.

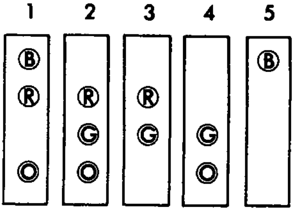
A. Can you construct a Solution which will fish out just card 2? (Hereafter, construct only multi-cube Solutions in response to the questions asked—even though in playing ON-SETS single-cube Solutions are permitted. This will provide more practice with multi-cube Solutions.)

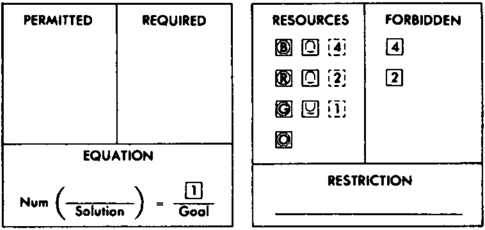
B. Can you construct a Solution that will fish out just card 3?

P2-A. B ∪ O ∩ G.

P2-B. No. It is impossible with the given Resources.

P3 (Problem 3)—Universe



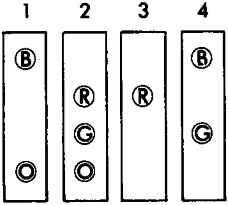


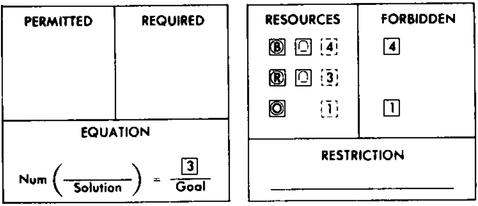
In the situation shown above, a Goal of 1 has been set. One Solution that would satisfy that Goal is [B] [∩] [R] [∩] [O]. It will draw out just card 1.

A. Construct a Solution that will draw out just card 2.

P3-A. [R] [∩] [G] [∩] [O].

P4 (Problem 4)—Universe



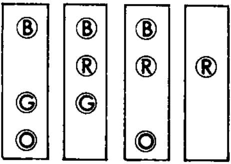


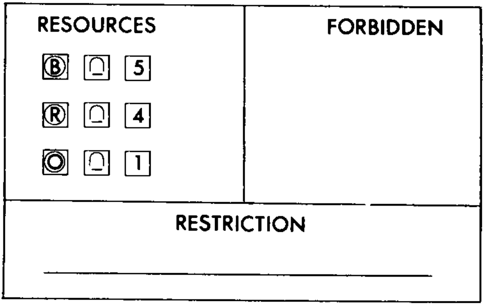
You are the Shaker and have set a Goal of 3 in the situation above. Player three exclaims: "*Challenge*—Goal impossible". Indeed, you Flubbed—you set a Goal that could not be satisfied with the cubes available and thus allowed player three to make a correct challenge. He wins and ends the round.

A. What Goal could you have set and what Solution would satisfy that Goal?

P4-A. 1, B ∩ O or R ∩ O.

P5 (Problem 5)—Universe



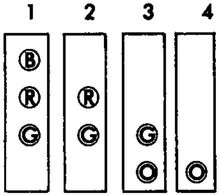


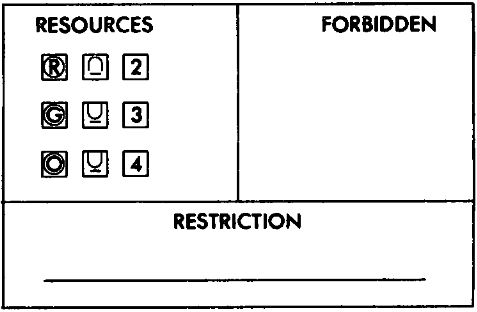
As the Shaker you are faced with the situation shown above. After examining the cards and cube faces, you conclude that no Goal can be set. You announce your finding to the other players by exclaiming "No Goal". However, another player challenges you and claims that it is possible to set a Goal.

A. Is the Challenger correct? Why?

P5-A. Yes. Num (R ∩ O) = 1.
Num (B ∩ R ∩ O) = 1.

P6 (Problem 6)—Universe





You are the Shaker in the situation shown above. You could set a Goal of 1 ([3] [Z]), 2, 3, or 4. List one more-than-one-cube Solution that could be constructed from this toss that would satisfy each of the following Goals and list the cards each Solution would draw out.

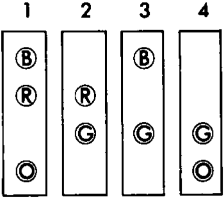
	Goals	*Solution*	*Cards*
A.	2		
B.	3		
C.	4		

From this point on parentheses will sometimes be used in Set-Names when their use prevents ambiguity. There are, of course, no parentheses in the game kit, so when cubes are played in an expression which may be interpreted in several different ways, separation of cubes may be used to indicate parentheses. Thus parentheses will be implied by spacing in the actual play of the game. E.g., the expression ([R] [∩] [B]) [∪] [◎] requires parentheses because the expression [R] [∩] ([B] [∪] [◎]) means something else. When playing the cubes to construct the first expression above, larger spaces would be left around the [∪] than around the [∩]—that is, [R] [∩] [B]   [∪]   [◎].

A. What would be the spacing to construct the second expression?

P6-A. [R] [∩] [G](1,2); [R]   [∩]   [G] [∪] [◎] (1,2).
P6-B. [R] [∪] [G](1,2,3); [R]   [∪]   [G] [∩] [◎](1,2,3).
P6-C. [R] [∪] [◎](1,2,3,4); [G] [∪] [◎] (1,2,3,4);
[R] [∪] [G] [∪] [◎](1,2,3,4).
A. [R]   [∩]   [B] [∪] [◎].

P7 (Problem 7)—Universe



Expressions may contain more than two color cubes and one connective, and in determining which cards such expressions fish out attention may need to be paid to more than just two colors. For example, the expression (R ∩ O) ∪ B will fish out all the cards that are BOTH red and orange and also those that are blue (Cards 1 and 3 above).

A. Construct an expression that will fish out, from the cards above, all those cards that are both red and green and also those that are orange.

B. Which of the above cards would such an expression fish out?

When a cube is placed in the Required section of the playing mat, that cube must be used in constructing the Solution. If he wants to avoid Flubbing, the player who moves a cube to the Required section must make sure that it is pos-

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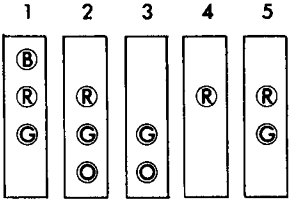
P7-A. (R ∩ G) ∪ O.

P7-B. 1, 2, 4.

sible to construct a Solution that includes *all* the cubes in the Required section after his play.

P8 (Problem 8)—Universe



PERMITTED	REQUIRED
	R

EQUATION

Num (Solution) = 2 1 / Goal

RESOURCES	FORBIDDEN
B ∪	4
R ∪	
G ∩	
O	

RESTRICTION

____

The Shaker has set a Goal of 3. You are the next player, and you have just moved the red cube to the Required section. You detected three different Solutions: B ∪ O, R ∩ G, and B ∪ (G ∩ O).

A. Which of the possible Solutions has your move eliminated?

P8-A. B ∪ O, B ∪ (G ∩ O).

When a player moves a cube to the Forbidden section, he is forbidding the use of *that* cube in constructing the Solution. He is not, however, prohibiting the use of other cubes bearing the same color or symbol. If a player wishes to avoid Flubbing, he must not forbid a cube unless it is possible to build a Solution that does not use that (or any other) Forbidden cube.

B. If the next player wishes to move a color cube to the Forbidden section, which cube (or cubes) may he move there without Flubbing?

C. Is there any other cube that he can safely move to the Forbidden section? If so, which one?

We now know that a player may either require or forbid the use of a particular cube but in making either move he must observe one very important precaution if he wants to avoid Flubbing (by violating the P-claim).

A. What is it that he must be so sure of?

When a player moves a cube on his turn from the Resources section to the Permitted section of the playing mat, he is permitting (but not requiring) the use of that cube in the Solution. Since a cube played into the Permitted section may

---

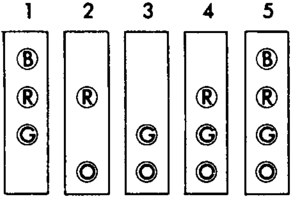
P8-B. [B] or [O].

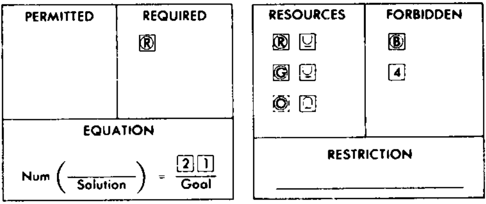
P8-C. Yes. [∪].

A. That a Solution ccn still be constructed by an appropriate allocation of the remaining resources.

be used in the Solution, a player to be safe must be careful that he does not inadvertently play a cube that will permit a Solution to be built with just one more cube from the Resources.

P9 (Problem 9)—Universe





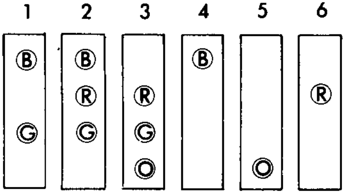
In the three-player game above, player 1 set a Goal of 3, player 2 required R, player 3 forbade B, and player 1 is now considering R ∩ G as a Solution. However, he believes that he cannot make a move to further his Solution without violating the A-claim by allowing a Solution to be built with one more cube from the Resources. Furthermore, he feels unsure of what the other players have planned. So, thinking it to be a noncommittal move, he moves O to the Permitted section. The second player challenges, claiming that player 1 has unnecessarily made it possible to build a Solution with one more cube from Resources. The Challenger is correct and wins the game.

A. How can a Solution be built with one cube from Resources?

B. What moves could player 1 have made which would not be Flubs?

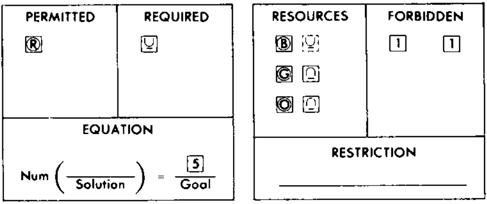
A player who, when it is his turn to play, fails to challenge when it is possible for him to do so correctly, has made a Flub and may be correctly challenged by one of the other players. Thus, if it is player 2's turn in the following situation:

P10 (Problem 10)—Universe



P9-A. Use $\cap$ to build R $\cap$ O.

P9-B. R, G, $\cup$, or O to Forbidden.
R or $\cup$ to Permitted or Required.



any play by him other than a challenge would be a Flub.

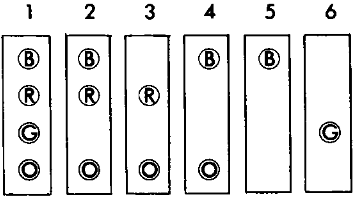
A. The Solution player 2 would then form is ______________ .

Suppose that play in another game has proceeded as follows:

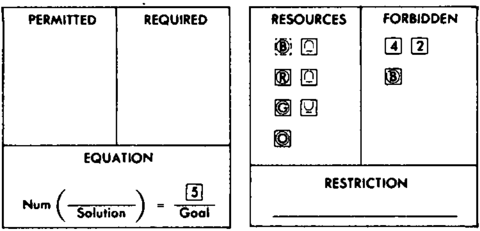
Play	*Player*	*Goal*	*Forbidden*	*Permitted*	*Required*
a	1	[5]	[4] [2]		
b	2		Ⓑ		

resulting in the following situation:

P11 (Problem 11)—Universe



P10-A. Ⓑ ∪ Ⓡ or Ⓡ ∪ Ⓑ.



Player 3 challenges, claiming that player 2 by forbidding the blue cube, has made construction of a Solution impossible.

A. Is player 3 correct in his contention that a Solution is impossible? Why?

B. Would a move of the green cube, instead of the blue cube, to the Forbidden section have been a Flub?

C. Indicate a possible Solution even after a [G] → F move (green to Forbidden move).

D. Which of the following moves would be Flubs as move b after the Goal of [5] had been set? Why?

P11-A. No. [G] [∪] [O] is a possible Solution.

P11-B. No.

P11-C. [B] [∪] [R] or [B] [∪] [O].

P11-D. b2. Building a Solution is made impossible, so it violates the P-claim.

Move

b1 [G] → P (green to Permitted)

b2 [∪] → F ([∪] to Forbidden)

b3 [∩] → R ([∩] to Required)

b4 [G] → R

4.2-C Complement of a Set

The two symbols discussed above connected two Set-Names to form a new Set-Name. The symbol [′] relates to only a single Set-Name, and it is placed to the right of that Set-Name. Thus, we can use it as follows:

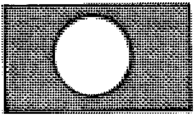
[R] [′]

or like this:

[R] [∪] [G]   [′]

where it is placed to the right of the Set-Name [R] [∪] [G].

The sign [′] added to the end of a given Set-Name forms a new Set-Name. The result is the name of a set that contains all the things that are in the Universe but are not in the set named by the given Set-Name. The new set is called the "complement" of the old set, and we can represent it by the shaded portion of the following diagram:



indicating that the complement of the set represented by the circle is everything outside the circle. Thus, the Set-Name

Ⓡ ′

is the name of the set of all cards in the Universe that are not red.

Exercises

P1 (Problem 1)

The set named Ⓑ ′ is the complement of the set named by Ⓑ.

A. Similarly, the set named by Ⓑ ∪ Ⓖ ′ is the ______ of the set named by Ⓑ ∪ Ⓖ. For short, we will say that (B∪G)′ is the complement of B∪G.

B. Likewise, (R∩O)′ is the complement of ______ .

P1-A. complement.
P1-B. R∩O.

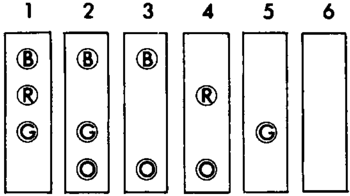
Note that B ∪ G   ′ is the name of a different set than that named by B   ∪   G ′. The cards fished out by B ∪ G   ′ are all those that are not either blue or green, while B   ∪   G ′ fishes out all those that are either blue or not green.

C. Write a Set-Name that would fish out all those cards that are red and not blue.

D. Write a Set-Name that would fish out all those cards that are not both red and blue.

Consider the following set of cards:

P2 (Problem 2)—Universe



The Set-Name B   ∪   O ′ would fish out cards 1, 2, 3, 5, and 6.

Which cards would the following Set-Names fish out?

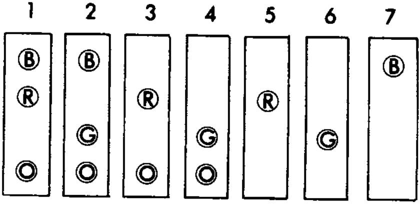
P1-C. R   ∩   B ′.
P1-D. R ∩ B   ′.

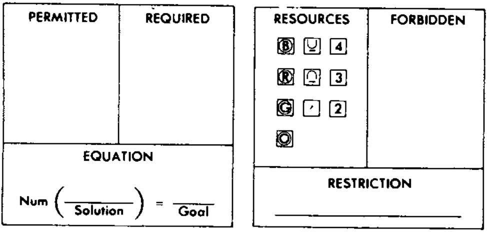
	Set-Name	Cards
A.	O ∪ B ′.	
B.	O ∩ B ′.	
C.	O ∩ B ′.	
D.	O ∪ B ′.	
E.	B ∪ O ′.	
F.	B ∪ O ′.	
G.	B ′ ∪ O.	
H.	R ∩ G ′	
J.	B ′ ∪ G ∩ R.	
K.	B ∩ O ′ ∪ G.	

Consider the following situation:

P2-A. 2, 3, 4, 5, 6.
P2-B. 4.
P2-C. 1, 4, 5, 6.
P2-D. 5, 6.
P2-E. 5, 6.
P2-F. 1, 2, 3, 5, 6.
P2-G. 2, 3, 4, 5, 6.
P2-H. 4.
P2-J. 1, 4.
P2-K. 1, 2, 4, 5, 6.

P3 (Problem 3)—Universe





If 2 is set as the Goal, there are at least 6 different Set-Names that will fish out cards 1 and 2 only to satisfy the Goal: B ∩ O, B ∩ G ∪ R, B ∩ R ′ ∪ O, B ∩ R ∪ O, B ∩ G ∪ O, and B ∩ G ′ ∪ O. There are, of course, many others also, which are simple variations of the 6 above, obtained by varying the order of the colors—for example: O ∩ B. There are 20 other pairs of cards that will satisfy the Goal, if a Set-Name for them only can be constructed.

For which of the following pairs of cards is a Set-Name possible? Give one example of a Set-Name for each pair for which a Set-Name is possible.

	Pairs of Cards	*Possible?*	*Example Expression*
A.	1, 3		
B.	1, 4		
C.	1, 5		
D.	1, 6		
E.	2, 3		
F.	2, 4		
G.	2, 5		
H.	2, 6		
I.	3, 4		
J.	3, 5		
K.	3, 6		
L.	4, 5		
M.	4, 6		
N.	5, 6		

P3-A. Yes. R ∩ O.
P3-B. No.
P3-C. Yes. R ∩ O ′ ∪ B.
P3-D. No.
P3-E. No.
P3-F. Yes. G ∩ O.
P3-G. No.
P3-H. No.
P3-I. Yes. B ′ ∩ O.
P3-J. Yes. B ∪ G ′.
P3-K. No.
P3-L. No.
P3-M. Yes. B ∪ R ′.
P3-N. Yes. B ∪ O ′.

4.2-D *Difference of Sets*

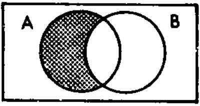
The connective [−] both looks like, and has a meaning that is similar to the meaning of, the minus sign in arithmetic. Like the cup and cap it is placed between two Set-Names to form a new Set-Name as in:

Ⓡ [−] Ⓑ

or

Ⓡ [∪] Ⓖ [−] Ⓑ

We can represent the difference of two sets, A − B, by the shaded portion of the following:



The minus sign is used to form the name of a set that contains all the things that are in the set named on its left *except* for those that are in the set named on its right. Thus, the Set-Name:

Ⓡ [−] Ⓑ

names a set that contains all the red cards except for those red cards that are also blue. (How many cards are in this set in the sample Universe shown on page 26?)

Notice that this time the order in which the Set-Names are placed around the connective is significant. When we used the cap, it didn't make any difference whether we played:

[Ⓡ] [∩] [Ⓑ]

or

[Ⓑ] [∩] [Ⓡ].

This is also true of the cup. However, with the minus sign, the set named by

[Ⓡ] [−] [Ⓑ]

is usually different from the set named by

[Ⓑ] [−] [Ⓡ].

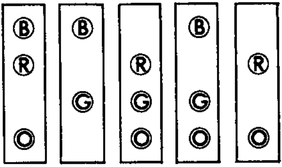
The set named by one of these two contains more cards than the other in the sample Universe. (Which one is larger?)

Exercises

From the following set of cards:

P1 (Problem 1)—Universe

1 2 3 4 5



the Set-Name [Ⓑ] [−] [Ⓡ] would fish out the set containing cards 2 and 4—that is, those blue cards that are not red (or alternatively expressed, the set of blue cards minus the set of red cards).

Which cards would the following Set-Names fish out?

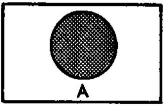
Set-Name	*Cards*
A. [Ⓡ] [−] [Ⓖ]	
B. [◎] [−] [Ⓡ]	
C. [◎] [−] [Ⓑ] [′]	
D. [Ⓖ] [∪] [Ⓡ] [−] [◎] [′]	
E. [Ⓡ] [−] [Ⓖ] [′]	

4.2-E Summary

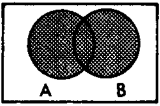
The meanings of the various symbols are summarized in the diagrams on the following page.

1A. 1, 5
1B. 4
1C. 1, 4
1D. 1, 3, 4, 5
1E 2, 3, 4

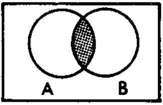
Meaning of Symbols



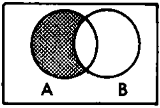
Set



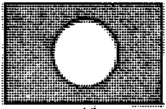
$A \cup B$
Union



$A \cap B$
Intersection



$A - B$
Difference



A'
Complement

4.3 The Blue-Symbol Cubes

4.3-A The Set-Names, $\underline{\vee}$ and $\underline{\wedge}$

Two of the symbols on the blue-symbol cubes are the names of particular sets. They have the same role as the dots on the color cubes. They can stand alone as Set-Names or be used in combination with the operations, [$\cup$], [$\cap$], ['], and [$-$], to form new Set-Names.

The [$\underline{\vee}$] denotes the universal set. This set contains every card in the Universe. If the first player has turned up six cards, then Num ([⊻]) = 6. The [⊻] combines in interesting ways with other Set-Names; for example the set named by [Ⓡ] [∩] [⊻] is always exactly the same as the set named by [Ⓡ].

A. What is a shorter name for the set named by [Ⓡ] [∪] [⊻]?

The [⊼] denotes the empty set (often called the null set). This set has no cards in it. No matter what cards are in the Universe Num ([⊼]) = 0.

One way to remember the difference between these symbols is to think of [⊻] as another kind of a cup that holds as much as possible and [⊼] as a kind of upside-down cup that holds as little as possible because everything has poured out.

Of course, [⊼] can be used in combination with other Set-Names and operations. For example the set named by [Ⓖ] – [⊼] is always the the same as the set named by [Ⓖ].

B. What is another (shorter) name for [⊼] [′]?

A. [⊻].

B. [⊼] [′] = [⊻].

4.3-B Relations Between Sets

On the blue-symbol cubes appear two symbols that denote *relations* between sets: one of these is [=] and the other is [⊆]. When one of these relations is placed between the names of two sets, the result is a statement about the sets.

In the advanced game (called Advanced ON-SETS), such a statement is called a *Restriction* and may serve as part of a Solution. When a Restriction Statement is made the Solution always has two parts: a Set-Name and one or more Restriction Statements. Such a Solution is called a C-Solution. (C stands for Combination Set-Name and Restriction Statement.)

Both Set-Names and Restrictions are rows of cubes, but Set-Names are names of sets of cards, while Restrictions are statements about sets. Set-Names are like the names "George" or "Mary" (which are names of people) while restrictions are like the sentences "George is fat" and "Mary is happy" (which are statements about people just as Restrictions are statements about sets of cards). Unlike names, Restrictions can be true or false.

Which of the following are Set-Names and which are statements?

A. [B] ['] [∩] [R] [∪] [G]

B. [R] ['] [∪] [G] [⊆] [B]

A. Set-Name
B. statement

C. [R] [′] [=] [V]

D. [R] [∪] [Λ]

E. [R] [∩] [Λ] [⊆] [B]

F. [B] [∪] [V] [=] [R] [′]

When a player builds a C-Solution all the cards in the Universe that make the Restriction Statement(s) false must be removed from the Universe. The Universe thus formed is called the restricted Universe. The Set-Name part of the C-Solution must then name a set which fishes out of the restricted Universe the number of cards equal to the Goal.

4.3-B1 Identity

One of the symbols used in making Restriction Statements is the [=]. Its use in expressing a relation between sets is similar to its use in expressing a relation between numbers. Statements about numbers that use this symbol are called "equations".

For instance, we may write:

$$5 + 2 = 3 + 4$$

to indicate that '5 + 2' and '3 + 4' are really different names of the same number. We might even write:

C. statement
D. Set-Name
E. statement
F. statement

Shakespeare = The author of Hamlet

to indicate that 'Shakespeare' and 'The author of Hamlet' are names of the same person. It is something like the latter that the [=] means in ON-SETS.

In building a Restriction, the [=] goes between two Set-Names* like this:

[B] [=] [G]

and it says that the two Set-Names on either side of the equal sign are names for the same set of cards; that the set named by [B] is identical to the set named by [G]. This statement may or may not be true for a given Universe. To make it true, one must remove all the cards that are in either one of the sets but not in the other. In the case of the above Restriction, one has to remove all the cards that are blue but not green or green but not blue, because the Restriction Statement would not be true about a Universe that contained such cards.

Remember that the expression [B] is not the name of a color but the name of a set of

---

*In the EQUATION section the '=' is placed between the names of numbers rather than between the names of sets of cards. The number on the right side of the '=' is named in the familiar way (that is, with a numeral), but the name of the number of the left side is formed by placing a Set-Name in the blank space of 'Num (__________)'. The number named by 'Num (Set-Name)' is the number of cards in the set named by the Set-Name that are contained in the Universe. An equation in the EQUATION section is true if and only if the names on both sides of the '=' name the same number.

cards. To say that $\boxed{B}$ $\boxed{=}$ $\boxed{G}$ is not to say that blue is the same color as green. As a matter of fact, this statement is not about colors at all. It is about sets of cards that are identified by the colors the cards have on them. Thus, this statement says that the two sets of cards named by $\boxed{B}$ and $\boxed{G}$ are the same set. We say that two sets are identical if they have exactly the same objects (cards) in them. In talking about sets we usually ignore the differences between the various names that we called the same sets ($\boxed{G}$ and $\boxed{B}$ in this case) and pay attention only to the things that are in the sets. This is why $\boxed{B}$ $\boxed{=}$ $\boxed{G}$ is true when both $\boxed{B}$ and $\boxed{G}$ are names of the same set of cards. This is just like saying:

$$3 + 4 = 5 + 2$$

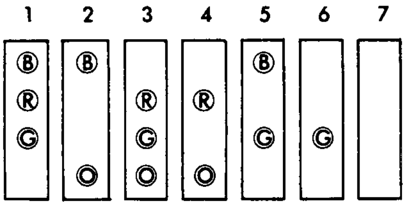
because both '3 + 4' and '5 + 2' name the number 7, even though the expressions '3 + 4' and '5 + 2' are not exactly the same expressions.

Exercises

P1 (Problem 1)

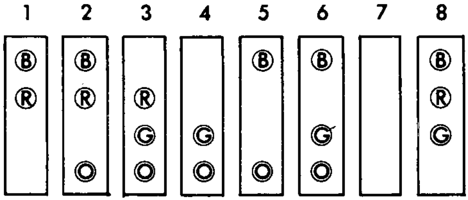
A. Of the Universe below, which cards must be removed to make the statement $\boxed{B}$ $\boxed{=}$ $\boxed{G}$ true?

P1-A. 2, 3, 6.



P2 (Problem 2)

From the following Universe:



list those cards that must be removed to make the following statements true:

A. [B] [=] [O]

B. [R] [=] [G]

C. [B] [=] [R]

* * *

P2-A. 1, 3, 4, 8.

P2-B. 1, 2, 4, 6.

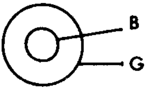
P2-C. 3, 5, 6.

4.3-B2 Inclusion

The $\boxed{\subseteq}$ also goes between two Set-Names like this:

$$\boxed{B}\ \boxed{\subseteq}\ \boxed{G}$$

This statement says that the set named on the left is included in the one named on the right, which we might represent pictorially like this:



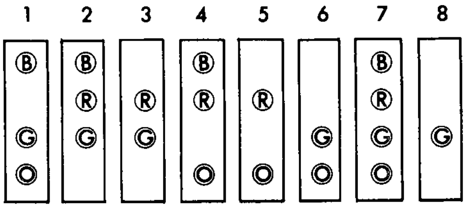
For this to be true we have to remove all the cards that are in the first set but not in the second. In this case we remove all cards that are blue but not green because these are the cards that would make the Restriction Statement false about the Universe.

$\boxed{B}\ \boxed{\subseteq}\ \boxed{G}$ is read as "Blue is included in Green" or as "Blue is a subset of Green."

Exercises

P1 (Problem 1)

Which of the following cards:



must be removed to make the following statements true?

A. R ⊆ G

B. B = O

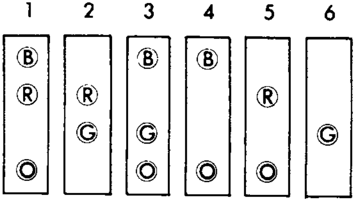
C. B ⊆ G

D. G ⊆ B

P1-A. 4, 5.
P1-B. 2, 5, 6.
P1-C. 4.
P1-D. 3, 6, 8.

P2 (Problem 2)

Which of the following cards:



must be removed to make the following statements true?

A. V ⊆ Ⓑ

B. Ⓑ ∪ Ⓖ = V

C. Ⓡ ∩ Ⓞ = V

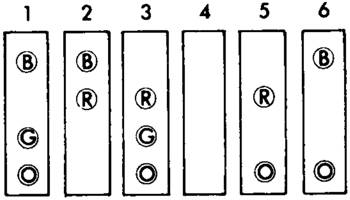
D. Ⓖ ′ = V

E. Ⓡ = V ′

P2-A. 2, 5, 6.
P2-B. 5.
P2-C. 2, 3, 4. 6.
P2-D. 2, 3, 6.
P2-E. 1, 2, 5.

P3 (Problem 3)

Which of the following cards:



must be removed to make the following statements true?

A. R ⊆ △

B. B ∪ G = △

C. R ∩ G ⊆ △

D. B ′ ⊆ △

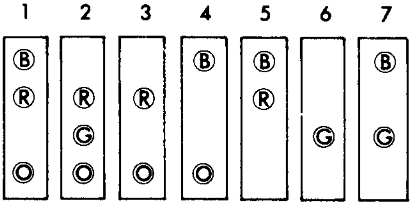
E. R ∩ V = △

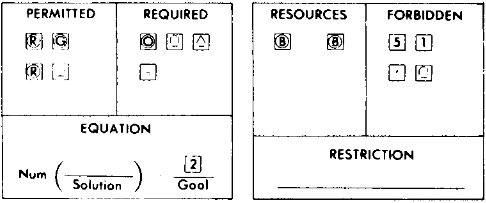
P3-A. 2, 3, 5.
P3-B. 1, 2, 3, 6.
P3-C. 3.
P3-D. 3, 4, 5.
P3-E. 2, 3, 5.

P4 (Problem 4)

Consider the following situation:

Universe





A challenge of the last move can correctly be made. That move violated either the A-claim or the C-claim, because it is now possible to build a C-Solution with one more cube from the Resources. The C-Solution Restriction is

Notice that this Restriction uses every cube from Required (as it must by game Rule 3.4′). This Restriction eliminates cards

A. ____, ____, ____, and _____ from the Universe. The Set-Name part of the C-Solution is

B. [ ] [ ] [ ] [ ] [ ] which fishes out cards

C. ____ and ____ .

The Set-Name (as it must by Rule 3.4′) uses every cube in Required except the [=].

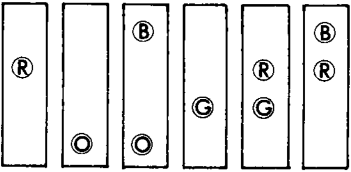
P4-A. 1, 2, 3, 4.

P4-B. [Ⓑ] [U̲] ([△̲] [∩̲] [◎]).

P4-C. 5, 7.

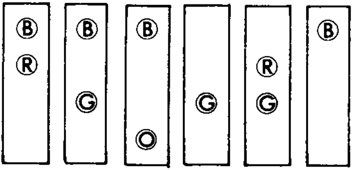
5. Types of Challenges with Examples of Each

a. Universe



Resources: R R R R R $\cup$ $\cup$ $'$ $'$ 2 2 4
Shaker says "No Goal".
Challenger says Shaker has Flubbed.
Burden of Proof: on Challenger; sustained by
Goal: [4] [2] (= 6) and
Solution: $R \cup R'$
Results: Challenger wins; Shaker loses.

b. Universe



Resources: B B R G O $\cap$ $\cap$ $-$ $-$
1 2 4
Shaker sets 1 as a Goal.

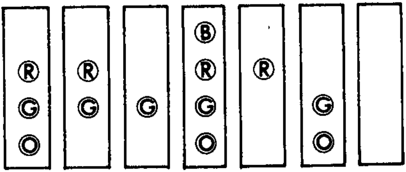
Challenger says Shaker has violated the A-claim.

Burden of Proof: on Challenger; not sustained.

The Challenger not only must build a Solution with one Resource cube (which he does with the orange cube), he must also prove that there was a possible Goal for which no one-cube Solution could be built. Since the Challenger cannot show such an alternative Goal, he has not sustained the burden of proof.

Result: Mover wins; Challenger loses.

c. Universe



Resources: O O R B G ∪ ∪ ∪ ∩
1 5 1

Shaker sets [1] [L] (= 0) as Goal. Challenger says Shaker has violated the P-claim.

Burden of Proof: On Mover; not sustained.

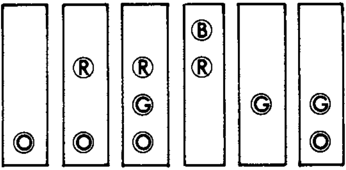
Result: Challenger wins; Mover loses.

In the remaining examples a game is in progress and the summary shows how the cubes have been placed on the mats. The most recently moved cube is circled. After the move shown by the circled cube, a challenge has been made. The type of challenge and its result are shown.

d. Universe: Same as in c.
Resources: B ′ ∪
Forbidden: B
Permitted: R ∩ ◎
Required: − G
Equation: Num ( ) = [3] [4] (= 7)
Challenge: A-claim violation.
Burden of Proof: on Challenger; sustained by (O − G)′.
Result: Challenger wins; Mover loses.

e. Universe Same as in c.
Resources: O B ∪
Forbidden: B ⊖
Permitted: R ′
Required: G ∪
Equation: Num ( ) = [3] [ε] (= 0)
Challenge: P-claim violation.
Burden of Proof: On Mover; not sustained.
Result: Challenger wins; Mover loses.

f. Universe



Resources: G G B R O

Forbidden: 2 ∪ ∩

Permitted:

Required: – ⊖

Equation: Num ( ) = [4] [ᔭ] (= 0)

Challenge: P-claim violation.

Burden of Proof: on Mover; sustained by (G – O) – G.

Result: Mover wins; Challenger loses.

g. Universe: Same as in f.

Resources: ′ ∩

Forbidden: 3 2 – B

Permitted: O G

Required: ′ ∪ R Ⓑ

Equation: Num ( ) = 5

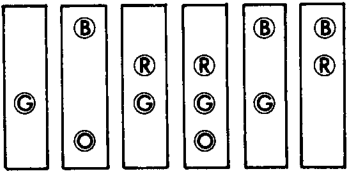
Challenge: C-claim violation stemming from an A-claim violation on some previous move.

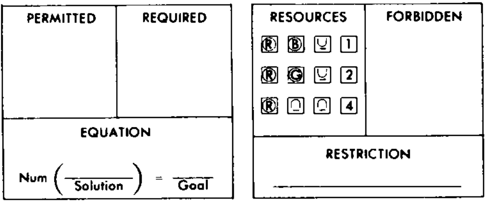
Burden of Proof: on Challenger; sustained by (R∪G)″.

Result: Challenger wins; Mover loses.

6. Some Sample Games of ON-SETS

a. Sample Game 1 (Basic) Universe

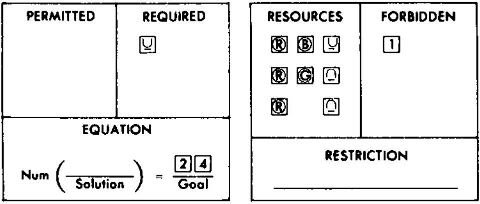




Player A has turned up a Universe of six cards. Player B has rolled the cubes. The symbols facing up are shown in Resources. Player B sets a Goal of 6 by placing the [2] and [4] side by side in the Goal section. He also puts the [1] into Forbidden since it can no longer be used. Player B is thinking of the Solution [G] [∪] [B].

Player C has the next turn and he puts a [∪] into Required. The Solution he has in mind is [B] [∪] [G].

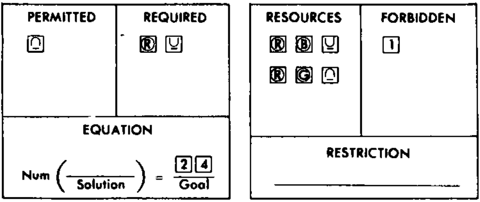
The playing mats now look like this:



Player A puts a [∩] into Permitted. He also is thinking of [B] [∪] [G] as a Solution.

Player B puts a [R] into Required. He is now thinking of [G] [∪] [B] [∪] [R] as a Solution.

The mats now look like this.



Player A challenges because he thinks Player B has violated the P-claim. Player C joins Player B. Players B and C have the burden of proof. Player B sustains it with the Solution (G∪B)∪R and Player C sustains it with (G∪B)∪(R∩R). Players B and C score 2 points each and Player A scores 0.

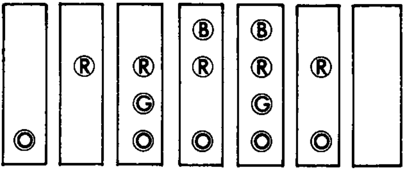
The above game can be described by the following summary:

Resources: R R R B G ∪ ∪ ∩ ∩
1 2 4

Play	*Player*	*Forbidden*	*Move Permitted*	*Required*	*Solution in Mind*	*Play that eliminates Solution*
a	B	1	Goal = [2][4]		G∪B	d
b	C			∪	B∪G	d
c	A		∩		B∪G	d
d	B			R	(G∪B)∪R	
e	A Challenges: P-claim violation. C Joins B. B's Solution: (G∪B)∪R. C's Solution: (G∪B)∪(R∩R). A scores 0, B scores 2, and C scores 2.					

The rest of the sample games will be presented in summary form. You will follow them more easily if you set up the game with cubes, cards and mats, and make the moves indicated.

b. Sample Game 2 (Basic)
Universe

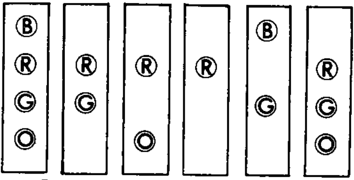


Resources: R R B B O ∩ ∪ ∪ −
3 5 5

Play	*Player*	*Forbidden*	*Move Permitted*	*Required*	*Solution in Mind*	*Play that eliminates Solution*
a	A	3	Goal: [5] [5] (= O)		B − O	b
b	B	∩ (bonus)		R	R − R	c
c	C	∪ (bonus)		∪	R − (R∪B)	e
d	A	B (bonus)	O		R − (R∪O)	
e	B	R			B − (R∪O)	
f	C			B	B − (R∪O)	

Player A now announces that a Solution can be built with one more cube from Resources. He correctly does *not* challenge Player C for violating the A-claim on move f, since Player C had no alternative move that would avoid allowing a Solution to be built with one more cube from the Resources and that did not violate the P-claim. All the players write down correct Solutions and each scores 1 point.

c. Sample Game 3 (Basic)
Universe

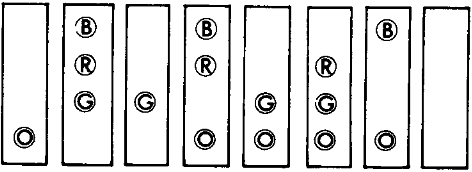


Resources: B B B R R $\cap$ $\cup$ $'$ $'$ 1 4 5

Play	Player	Forbidden	Move Permitted	Required	Solution in Mind	Play that eliminates Solution
a	A	4, 5	Goal: 1		$B \cap R$	
b	B		R		$B \cap R$	

c C Challenges: A-claim violation.
A Joins C.
C's Solution: R'.
A's Solution: R'.
A scores 1, B scores 0, and C scores 2.

d. Sample Game 4 (Advanced)
Universe



Resources: R B B B O O O G $\triangle$ $=$
$\subseteq$ $-$ $\cap$ $\cup$ $'$ 1 3 4

Play	Player	Forbidden	Move Permitted	Required	Solution in Mind Restriction(s)	Solution in Mind Set-Name	Play that eliminates Solution
a	A	3, 4	Goal: 1			B − R	b
b	B			⊆	B⊆G	B	d
c	C		∧		O⊆∧	B	d
d	A			−	∧⊆B − R	B − R	
					A revises his original Solution by building a Restriction that does not restrict.		
e	B	O			B − ∧⊆G	B − ∧	
f	C		O		O⊆∧ − B	B − O	
					B and C revise their original Solutions. C makes a deliberate Flub. He hopes to trap A.		
g	A		∩		∧⊆B − R	B − R	

h C Challenges: C-claim violation that stems from an A-claim violation.
B Joins A.
C's Solution: O⊆∧ − B; B − O.
A scores 0, B scores 0, and C scores 2.

e. Sample Game 5 (Advanced)

Same Universe as in d.

Resources: R R O O B B G G
∪ ∩ ∩ ′ ⊆ ∨ = 1 4 5

Play	Player	Forbidden	Move Permitted	Required	Solution in Mind Restriction(s)	Solution in Mind Set-Name	Play that eliminates Solution
a	A	4	Goal = [5] [1]		R⊆∨	R∪O	c
b	B		=		B = R	∨	
c	C			B		B∪O	
d	A	⊆R			B = B	B∪G	
e	B		∪		B = R	∨∪B	

f C Challenges: A-claim violation.
A Joins B.
C's Solution: B∪O.
A scores 0, B scores 0, and C scores 2.

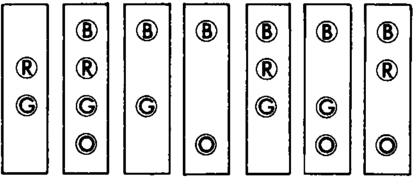
In this game both A and B seem to have forgotten that as long as neither = nor ⊆ is in Required, a Simple (Set-Name) Solution can be built.

7. Puzzles

7.1 One-Cube-Solution Puzzles

In these puzzles a summary is presented of an ON-SETS game in progress. The puzzle is to find a Solution which uses all the cubes in Required, any of those in Permitted that are needed, none from Forbidden, and at most one cube from Resources.

Universe



A. Basic ON-SETS

Resources: G O

Forbidden: G $\cap$

Permitted: R B

Required: $'$ $-$ $\cup$

Equation: Num ( ) = $\frac{\boxed{3}}{\boxed{2}}\boxed{1}$ (=7)

B. Advanced ON-SETS

Resources: $\cap$ $\cap$ $\cup$ B R O

Forbidden: 2 3 B $\cup$ R

Permitted: $\vee$ B R G

Required: $=$ $\wedge$

Equation: Num ( ) = $\boxed{1}$

Suggested Solutions:

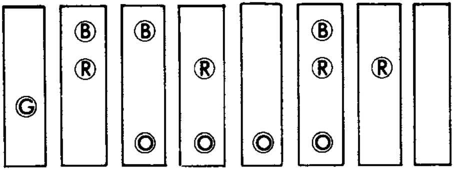
A. $(G - (R \cup B))'$

B. $B = \Lambda$; $\Lambda \cup V$

7.2 CAP-Claim Puzzles

A summary is presented of an ON-SETS game in progress. The most-recently-moved cube is circled. The puzzle is to indicate a claim that is violated by the move of the circled cube and sustain the burden of proof if Challenger has it.

Universe



A. Basic ON-SETS

Resources: G B $\cap$

Forbidden: 3 2 $\cap$

Permitted: O R Ⓑ

Required: $-$ $\cap$

Equation: Num ( ) = $\boxed{5}$

B. Advanced ON-SETS

Resources: B B R G $\cup$

Forbidden: R $-$ ⊜

Permitted: 4 3 $\vee$ B R O

Required: $\subseteq$ $\cap$ $-$

Equation: Num ( ) = $\boxed{0}$

Suggested Solutions:

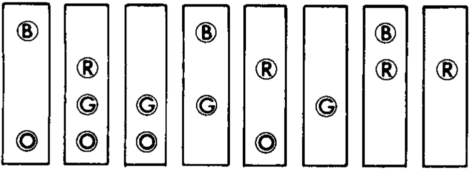
A. The move violates the P-claim. The Goal was impossible and hence every subsequent move was a Flub. The move also violates the C-claim stemming from a P-claim violation.

B. The move violates the C-claim stemming from an A-claim violation. The C-Solution that could have been built was: $(B \cap R) - O \subseteq G$; $(B \cap R) - O$.

7.3 Taboo-Move Puzzles

An ON-SETS game is shown in progress. The puzzle is to list all the moves that are Flubs, to tell which claim they violate, and to give Solutions for A-claim violations.

Universe



A. Basic ON-SETS

Resources: B R − ∩

Forbidden: 3 2 G

Permitted: ∪ G

Required: ′ O

Equation: Num ( ) = [7]

B. Advanced ON-SETS

Resources: $\cup$ R G B O $\cap$

Forbidden: B $\cap$ O R 4 3

Permitted: $\vee$ $\subseteq$

Required: $=$ $-$ G

Equation: Num ( ) = [1]

Suggested Solutions:

A.

Taboo Move	*Claim Violated*	*Solution*
$\cap$ to Permitted or Required	A	$(B \cap O)'$
B to Permitted or Required	A	$(B \cap O)'$
$-$ to Permitted or Required	A	$[O - (G \cup R)]'$
R to Permitted or Required	A	$[O - (G \cup R)]'$

B.

Taboo Move	*Claim Violated*	*Solution Restriction*	*Solution Set-Name*
G to Permitted	A	$G = R - G$	$\vee - G$
R to Permitted or Required	A	$G - R = O$	$G - R$
B to Permitted or Required	A	$B - G = R$	$B - G$
O to Permitted or Required	A	$O - G = R$	$O - G$
$\cup$ to Permitted	A	$G = R - \vee$	$\vee - (G \cup R)$
$\cap$ to Permitted	A	$R = G - \vee$	$\vee - G$
G, R, B, O, $\cup$, or $\cap$ to Forbidden, Permitted or Required	C stemming from A	$R = G - \vee$	$\vee - G$

8. Introductory Games

This section describes some games that can be used to introduce the fundamental ideas of set theory more slowly, using games that are somewhat simpler to play than the Basic and Advanced Games. These simpler games may be useful for those who have difficulty with the Basic or Advanced Games or they may be useful for the classroom use of ON-SETS. Beginning at a level simple enough for kindergarten students, these introductory games gradually increase in complexity. At the end of this section a few games of solitaire are also described.

The games described in this section fit into four basic categories, the FISH games, the OUT games, the SQUAD games and the SETS games. All the games in each of these categories have similar structures as games, but they deal with different parts of set theory. Thus, there is a FISH game that gives practice with single-cube Set-Names, a FISH game that gives practice with each of the connectives [∪], [∩], [−] and [′], and finally, FISH games that give practice in making statements. These are given the natural names: CUBE FISH, CUP FISH, CAP FISH, ...

In all of these games except for those that use [′], the blank card should be left out of the deck.

8.1 Cube Games

8.1-A CUBE FISH

Aim of game: To teach the notion of a set (of cards) defined by a property (color).

Number of players: 2 or more.

Equipment: The deck of cards with the blank card left out and one of the color cubes.

Aim of players: To get the largest number of cards.

Method of play: The first player places six cards face up on the table so that all the other players can see them. The player to the left of the first player then rolls the cube and takes out all the cards that have that color on them that shows face-up on the cube. These are the cards that the cube "fishes out", and he places these cards face down in front of him. The first player now deals out enough additional cards face up on the table so that there are again six cards showing. The player to the left of the last cube-thrower now throws the cube and each player proceeds in turn and in the same manner until the dealer no longer has any more cards left to deal out. At this point play proceeds until there are no more cards left on the table.

Scoring: Each player counts up the number of cards that he has face down in front of him at the end of play. The player with the largest number of cards in front of him wins. If there is a tie, the players with the highest score may throw a numeral cube to break it.

Alternate method of scoring: Each player counts up the number of cards that he has fished out for that round and that is his score for that round. The deal moves to the player to the left of the player who dealt in the preceding round. The first player to get a score of 31 points at the end of a round wins. Again, in the case of a tie, a numeral cube can be used to break it.

8.1-B OUT CUBE

Aim of the game: To give further practice with the notion of a set as defined by a property.

Number of players: 2.

Equipment: The deck of cards with the blank card left out and one of the color cubes.

Aim of the player: To get rid of all the cards in front of him.

Method of play: One of the players mixes up the cards and turns up seven cards in front of his opponent and then seven cards (also face up) in front of himself. The extra card is placed face down in front of the dealer.

The players then proceed to throw the color cube in turn, starting with the player who did not deal. After each throw, the player who made the throw removes all his cards that have the color showing on the top face of the cube and turns them face down in a pile to the side.

The first player who has turned all of his cards face down wins.

8.1-C CUBE SQUAD

Aim of the game: To give more practice with the notion of a set as defined by a property.

Number of players: Any number can play, but there should be one deck of cards for each team (or squad) of players.*

The blank card should be left out of all decks.

Equipment: As many decks of cards as there are teams and one color cube.

Aim of the team: To get rid of all its cards.

Method of play: Each team places six cards face up in front of it. The teacher (or a non-playing student) throws the color cube and announces the color that comes face up. Each team of players removes those cards from its piles that have that color on them and places them face down to the side. They then deal out enough cards so that there are again six cards face up in front of them, and the cube is thrown again. When they no longer have any cards to deal out, they proceed without dealing out new cards. The first team that gets rid of all the cards in front of it, either because its whole deck of cards has been used up or because all six cards in front of it were removed by one

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*This game is intended mainly for a classroom that has several game kits.

throw, wins that round. In case of a tie, the winning teams can have a "sudden-death" play-off round with six cards each. The first team to get more cards out on a roll is the winner.

8.1-D CUBE SETS

Aim of the game: To introduce the basic structure of the game ON-SETS and give further practice with the notion of a set (of cards) as defined by a property (color on the face of the cube).

Number of players: 2 or more.

Equipment: A playing mat, three color cubes, three numeral cubes and a deck of cards with the blank card left out.

Aim of the players: To get the largest number of cards.

Method of play: The first player deals out six cards face up on the table. The player to his left throws the numeral cubes and the color cubes. If he can find a color cube that fishes out precisely the number of cards that appears on one of the numeral cubes, he places the numeral cube in the EQUATION section over the word GOAL and the color cube on the line to the left of the '=' sign. (If there is more than one such combination he may play any correct combination.) If the player is correct, he gets those cards fished out by the expression and turns them face down in front of him. Play then proceeds with this player dealing out new cards to replace the ones he fished out and the player to his left rolling the cubes. Play ends when there are no more cards on the table and no more cards left to deal out. The winner is the player with the largest number of cards at the end. Again, ties can be settled by six-card sudden-death play-offs between the winners with each one rolling a cube.

8.2 Cup Games

8.2-A CUP FISH

Aim of the game: To give practice with the use of the cup $\boxed{\cup}$.

Number of players: 2 or more.

Equipment: The deck of cards with the blank card left out, one of the red-symbol cubes, and two of the color cubes.

Aim of the players: To get the largest number of cards.

Method of play: The red-symbol cube is placed on the table with the cup face up. Play proceeds as in CUBE FISH except that two cubes are rolled and cards are now fished out by a Set-Name made by placing one color cube on either side of the cup. This Set-Name is interpreted as described in section 4.2-A.

8.2-B OUT CUP

Aim of the game: To give further practice with the use of the cup.

Number of Players: 2.

Equipment: The deck of cards with the blank card left out, two color cubes and one red-symbol cube.

Aim of the players: To get rid of all the cards in front of them.

Method of play: The red-symbol cube is placed on the table with the cup face up. Play proceeds as in OUT CUBE except that two cubes are rolled and cards are now fished out by a Set-Name made by placing one color cube on either side of the cup. The resulting Set-Name is interpreted as described in section 4.2-A.

8.2-C CUP SQUAD

Aim of the game: To give further practice with the use of the cup.

Number of players: 2 or more.

Equipment: A deck of cards for each team, two color cubes and one red-symbol cube. The blank card should be left out of all decks.

Method of play: Play proceeds as in CUBE SQUAD except that the red-symbol cube is placed with the cup face up by the Shaker who then throws only the two color cubes. He places these cubes on either side of the cup (without turning them over) and announces this Set-Name. The Set-Name is interpreted as described in section 4.2-A. The cards fished out by the Set-Name are turned face down by each team, and play proceeds as in CUBE SQUAD.

8.2-D CUP SETS

Aim of the game: To introduce several new features of the Basic Game; and to give further practice with the use of the cup.

Equipment: Four color cubes, three numeral cubes, one red-symbol cube and the deck of cards with the blank card left out.

Method of play: Same as CUBE SETS except that now a red-symbol cube is placed into the EQUATION section of the playing mat on the line to the left of the '=' sign with the cup face up. The player to left of the dealer throws the numeral and color cubes. If he can find two color cubes that when placed around the cup fish out exactly the number of cards that appears on one of the numeral cubes, he places the numeral cube in the Goal section and the color cubes around the cup in the Solution Section. (If there is more than one such combination he may play any correct combination.) If the player is correct, he gets those cards fished out by the Set-Name and turns them face down in front of him. Play then proceeds with this player dealing out new cards to replace the ones he fished out and the player to his left rolling the cubes. Play ends when there are no more cards on the table and no more cards left to deal out. The winner is the player with the largest number of cards at the end.

8.3 The Cap, Diff, and Comp Games

Each of the Cup games can be played with the red-symbol cubes placed with a different face up. If the cap is played face-up, the games are called CAP FISH, OUT CAP, CAP SQUAD and CAP SETS; if the set-difference sign is placed face-up, the games are called DIFF FISH, OUT DIFF, DIFF SQUAD and DIFF SETS; if the $\boxed{'}$ is placed face-up, the games are called COMP FISH, OUT COMP, COMP SQUAD, and COMP SETS. In the Comp games the blank card should be left in the deck.

8.4 The Statement Games

The statement games are played just like the cup games except:

1. Either $\boxed{=}$ or $\boxed{\subseteq}$ is placed face up and a shaking set (three red-symbol cubes and four color cubes) is thrown.

2. The cards that belong to the largest set of cards for which the statement formed by one of the players is true are the cards fished out. These sets are called the "truth sets" of the statements.

3. The games are called EQ (pronounced "eke") FISH, EQ OUT, EQ SQUAD and EQ SETS if the equal sign is used and INC (pronounced "ink") FISH, INC OUT, INC SQUAD and INK SETS if the $\boxed{\subseteq}$ is used.

4. In the SQUAD games, the Shaker may build any statement that he or she wishes to build.

8.5 Solitaire Games for the Lone Player

8.5-A ONE

Aim of the player: To devise a Set-Name that fishes out exactly one card.

Materials: All the cards and a shaking set from either the Basic or Advanced Games, without the numeral cubes.

Method of play: The player deals out all the cards face up in front of him. He then throws the cubes. He wins if he can devise a Set-Name, or a Set-Name together with statements, that fishes out exactly one card.

8.5-B GO-THROUGH

Aim of the player: To "go through" the whole deck of cards.

Materials: A deck of cards, three numeral cubes, three color cubes and two red-symbol cubes.

Method of play: The player deals out six cards face up in front of him, and places the remaining cards off to one side. He then throws all eight cubes. He places the numeral cubes off to one side and attempts to construct a Set-Name using the symbol and color cubes (he need not use all of them) that fishes out exactly the number of cards indicated on the face of one of the numeral cubes. He places the cards that are fished out face down in front of him. He deals out enough cards from the stack he placed to one side to replace the cards that have been removed, tosses the cubes, and forms a Set-Name. He continues in this manner until the deck he is dealing from is exhausted. Play terminates either when an appropriate Set-Name cannot be formed or when the player has "gone through" the whole deck of cards so that all of them are now lying face down in front of him. He wins only in the latter case.

8.5-C *A-B*

Aim of the player: To devise a given number of Set-Names that satisfy a given goal.

Materials: All the cards, all the cubes except for one of the numeral cubes.

Method of play: The player deals out all the cards face up on the table and throws the cubes. Let us call the numerals showing face up on the numeral cubes "A" and "B". If the player can make A Set-Names that extract B cards or B Set-Names that extract A cards, all at the same time, he wins.

9. The Idea of a Set

The idea of a set is one of the basic ideas of mathematics. A *set* is any collection of objects. For example, the set of all the states in the United States is a set, and among the things that are in this set are Hawaii, New York and Tennessee. There are small sets such as the set that contains only William Shakespeare; larger sets such as the set that contains all the automobiles in the United States, and even larger sets such as the set that contains all the positive whole numbers (1, 2, 3, and so forth).

Bringing things together into a set is a mental operation and it does not necessarily imply that these things have to have anything in common or that they have to be near each other. The Eiffel Tower, the Queen of Spain's nose and the planet Pluto can make up a set, while Mickey Mouse and Miami Beach can make up another.

We speak of the things that are in a set as the *elements* or *members* of that set. A set is not the same as its elements. Thus, although Memphis is in Tennessee, it is not in the set of all the states in the United States. (If it were, that set would have 51 members rather than 50 because it would contain the fifty states and the city of Memphis.)

We can specify a set in two different ways. We can name each of the things in it. We did this when we talked about the set that contained William Shakespeare, and we could have done this when we talked about the set of the states in the U.S. although that would have been a long list. However, we could not have done this, even in principle, when we wanted to talk about the set that contained all the positive whole numbers because the list would be infinitely long, so that we would never finish writing it out.

The second way of specifying a set is to state a property that things have to have to be members of it.* This is how we defined the set of all the states in the U.S. and it is how we have to specify an infinite set. A property can be looked at as a sentence with a hole in it, like:

. . . is a state in the U.S.

where the three dots indicate the hole.

We can tell whether or not something is a member of the set of things that has this property by putting the name of the thing we want to check into the hole and seeing if the resulting sentence is

*We said above that things do not have to have a property in common to be in a set. This is true if you think of a property as we usually think of it, but if we permit "artificial" properties, then every two things have at least one property in common that no other thing has. The property that Mickey Mouse and Miami Beach have in common is the property that we say a thing has by:

. . . is either Mickey Mouse or Miami Beach.

This is not as "natural" as the property that we say a thing has by:

. . . is red.

But it is a property nevertheless.

true. Thus, we put the word 'Tennessee' into the above and get:

Tennessee is a state in the U.S.

The sentence is true, so Tennessee is in that set. However, if we put the word 'Memphis' in that hole, we get:

Memphis is a state in the U.S.

and that sentence is not true, so Memphis is not in that set.

Two sets can be the same set even if they are defined differently. Thus, the set that has William Shakespeare as its only element is the same as the set that has the author of *Hamlet* as its only element.

Notice what this implies. Suppose that we write the name of a set by putting the names of the things in it between braces like this:

{C, A, T}

with commas in between the different names (so that we know that we have a set with three elements and not the set with a single cat in it). This is the set, in other words, that has as its only three members the letters C, A and T. This is the same as the set:

{A, C, T}

or the set

{T, C, A}

because, although the order of the elements is different, these three sets contain exactly the same elements and therefore are the same set.

There are differences between the name of a set, its elements and the set itself. In ON-SETS, the *names* of sets are formed with the *cubes*, and the *members* of these sets are *cards*. The sets themselves never really appear on the table at all, since they are abstractions. You may think about them, but you never really see them (although you may see all their elements).

ON-SETS is largely concerned with the relationship between the name of a set and the number of things that are in the set named. The color cubes name sets in terms of the property that a thing must have to be in the set. These properties, however, are not properties of the set. They are properties of the *elements* in the set. The set of red cards is named by the red cube-face, but that set has no color at all (since it is an abstraction). If no red cards are turned up on the table, the set of all red cards still exists, but in this game it has no elements. The color cubes, in other words, are names of collections (or sets) of cards, just as 'The Citizens of Memphis' and 'The Citizens of Atlantis' may be the names of collections of people. Just as there is a difference between the name 'Sam' (which begins with an S) and the person Sam (who begins at the top of his head), so there is a difference between a set (a collection of cards) and its name (a row of cubes).

In constructing expressions in ON-SETS, one takes the names of sets and puts them together, using symbols like cup ($\cup$) and cap ($\cap$) to form new names of sets. If one places a cup between the names of two sets, say,

the set of all even numbers

and

the set of all odd numbers

one gets the name of the set that contains everything that is in either or both of these sets. Thus the expression

the set of all even numbers $\cup$ the set of all odd numbers

is the name of the set of all whole numbers.* And the expression

the set of all even numbers $\cup$ the set of all whole numbers

is another name of the same set.

The cap ($\cap$) is placed between the names of two sets to form the name of a new set that contains only the elements that are in *both* of the sets named by the expressions written on either of its sides. Thus,

the set of all even numbers $\cap$ the set of all odd numbers

*We are assuming that 0 is an even number since it is divisible by two without a remainder.

names the set that has no members at all (since no number is both odd and even), while the expression

$$\text{the set of all even numbers} \cap \text{the set of all whole numbers}$$

names the set of all even numbers.

We use the minus sign (−) to form the name of the set created by taking away the members of one set from another. Thus,

$$\text{the set of all whole numbers} - \text{the set of all even numbers}$$

is the name of the set of all odd numbers, while

$$\text{the set of all even numbers} - \text{the set of all whole numbers}$$

is the name of the set that has no members at all. As a matter of fact, it is another name for the set that we previously called

$$\text{the set of all even numbers} \cap \text{the set of all odd numbers.}$$

This set has a lot of names, but no members. To say that this set is only *one* set is a convention agreed upon because it makes the theory simpler. Our definition of what it means for two sets to be alike really only covers the case where sets have members, but it would be easy to rephrase it so that it covered this case too. We could say that two sets are *not* identical only if one of them has a member that the other does not.

Because this set without any members appears so often, it is given several special names. It is called "the empty set" or "the null set", or it is called "$\wedge$", which is a name for it just as 'Sam' can be the name of a man.*

Just like the null set, Sam, the man, can have several names. We may refer to him as

the man who lives in the gray house

or as

my bald uncle

or by a variety of other names. But he is always the same man.

There is another set that has a particular name. It is called "the universal set" or "$\vee$". This is the set that has everything in the Universe of Discourse as a member of it.

The prime sign (′) can be placed to the right of the name of any set to form the name of the set that contains everything that is not in the set named. Thus by placing a prime sign to the right of the name

the set of all numbers

we get

(the set of all numbers)′

which names the set of all things that are not numbers, and by placing it to the right of

$\vee$

*The empty set is frequently denoted by the symbol 'ϕ,' or by other names.

we get

$$V'$$

which names the set of all things that are not in the universal set. (This set has a special name. What is it?)

Using only this vocabulary, all we can do is call sets by their names. It is as though our vocabulary consisted only of words like 'George', 'Sam', 'Mary', and other names. The symbols '$=$' and '$\subseteq$' serve as verbs and permit us to make statements about sets, just as adding verbs like 'loves' permits us to moke statements about people such as

George loves Mary.

You are probably familiar with the '$=$' sign from arithmetic. When we write

$$2 + 2 = 4,$$

we are saying that

$$2 + 2$$

and

$$4$$

are names of the same number.

When we write

The set of even numbers	$\cup$	the set of odd numbers	$=$	the set of whole numbers,

we are saying that

the set of even numbers	$\cup$	the set of odd numbers

and

the set of all whole numbers

are really names of the same set. This is very much like the phrases

The author of "Hamlet"

and

The author of "Macbeth"

which name the same person.

The symbol '$\subseteq$' might be read as "is included in" or "is a subset of". When we place this sign between the names of two sets we make the statement that one set is included in the other. Thus

The set of all even numbers	$\subseteq$	the set of all whole numbers

says that the set of all even numbers is included in the set of all whole numbers. This is true because everything that is in the first set (even numbers) is also in the second set.

Sets are always included in themselves; so the following is true:

The set of all even numbers	$\subseteq$	the set of all even numbers.

Statements need not be true, of course. It is perfectly all right to write

$$\text{The set of all whole numbers} \subseteq \text{the set of all even numbers}$$

even though the statement is false. It is false because there are numbers in the first of these sets (1, 3, 5, 7, . . .) that are not in the second set. Statements can be true or false. Names of sets, although they are used in making statements about sets, are neither true nor false.

10. The Uses of Sets

So what is the point of all this?

It is probably not an exaggeration to say that all of mathematics is the study of sets of one kind or another (though it is usually not the study of sets of cards). Because of this, the language of set theory (or the theory of sets) is the basic language of mathematics and mathematics, in turn, is the basic language of science.

Numbers, for example, are properties of sets, not of things. For instance, consider the following picture:

What number does it represent? Does it represent the number 1 (since there is one hand there) or does it represent the number 5 (because there are five fingers there)?

This example suggests that numbers are not properties of physical things. What then are they? Perhaps they are marks on paper like

2, 4, 5

and the like. But that is not plausible either. Where was the symbol '5' in the drawing above (and yet there were five fingers)? When the Romans wanted to say that there were six things they used the symbol 'VI', and yet it is hard to imagine that they were talking about a different number just because the symbol 'VI' is different from the symbol '6'.

Actually symbols like '2', '3', '4', etc., are not numbers, but the names of numbers. Numbers are sets whose members are themselves sets. It is these sets of sets that the numerals like '6' and 'VI' name (just as cubes name sets of cards). It is not fingers, but the set of all fingers on a hand, to which we attribute the number five. It is the set of all hands, and not the hand itself, in that picture above to which we attribute the number one.

Numbers are special kinds of sets. They are sets whose elements are themselves sets. Thus the number five is a set, one of whose members is the set of all fingers in that picture. However, a full explanation of what kinds of sets numbers really are is beyond the scope of this introduction, and the reader who wants to pursue this matter should turn elsewhere (say to an introductory book on set theory).

Other things that mathematicians study are also sets. Euclidean space (the kind one studies in geometry) is a set, and so are the triangles, the lines, and other things that one finds in this space. The addition sign in arithmetic is the name of a set, and so is the integration sign in calculus.

Because mathematics is a study of sets, the language of set theory has become the basic language of modern mathematics. Those in different branches of mathematics study different kinds of sets, but they all use the same underlying language to talk about the sets they study. Some of the basic terms of this language are what ON-SETS is intended to teach. In other words, ON-SETS can be looked at as an introductory course in how to "speak mathematics". It is a long step from "speaking basic mathematics" to doing mathematics (although what one does in playing ON-SETS more nearly resembles what mathematicians do than memorizing the multiplication table does), just as speaking Japanese is a long way from writing (good) poetry in Japanese. But it is a first step.

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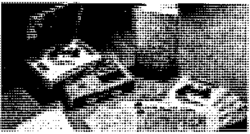

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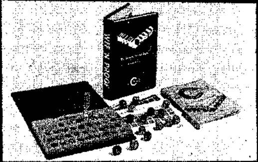

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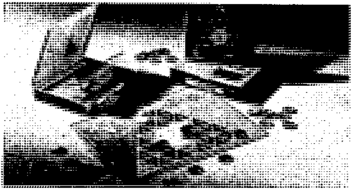

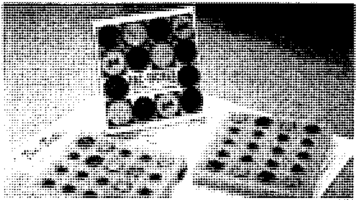

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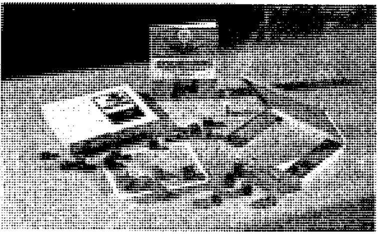

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