Lecture 9: Topology

Topology studies properties of sets which do not change under continuous reversible deformations. For a topologist, a coffee cup with one handle as the "same thing" as a doughnut. One can deform one into the other without punching any holes or ripping things apart. Similarly, a plate and a croissant are the same. A plate and a coffee cup are different topologically. On a plate for example, any closed curve can be deformed continuously to a point while on a coffee cup, there are closed curves going around the handle which can not be deformed to a point. For a topologist the letters O and P are the same but different from the letter B. Can can deform P into O but not produce P from B without ripping something or filling a hole. The modern mathematical setup is beautiful: a **topological space** is a set X with a set O of subsets of X containing both $\emptyset$ and X such that finite intersections and arbitrary unions of sets in O are in O. The sets in O are called **open sets**, the collection of open sets O is called a **topology** and the complement of open sets are called **closed**. Examples of topologies are the trivial topology $O = \{\emptyset, X\}$ where no open sets besides the empty set and X exist or the discrete topology $O = \{A \subset X\}$, where every subset is open. But these are in general not interesting. A familiar example is to let O be the collection of sets U in the plane X for which every point is the center of a small disc contained in U. An important class of topological spaces are **metric spaces**, where a set X is equipped with a **distance function** $d(x,y) = d(y,x) \geq 0$ which satisfies the **triangle inequality** $d(x,y) + d(y,z) \geq d(x,z)$ and for which $d(x,y) = 0$ if and only if $x = y$. A set U in a metric space is open if to every x in U, there is **ball** $B_r(x) = \{y | d(x,y) < r\}$ of positive radius r contained in U. Metric spaces are topological spaces but not all topological spaces are metric: the trivial topology for example is not. If we want to do **calculus** on a topological space X, each point has a neighborhood called **chart** which is topologically equivalent to a disc in Euclidean space. Finitely many such neighborhoods covering X form an **atlas** of X. If they fit nicely together, one obtains a **manifold**. Two dimensional examples are the **sphere**, the **torus**, the projective plane or the **Klein bottle**. Topological spaces X, Y are called **homeomorphic** meaning "topologically equivalent" if there is an invertible map from X to Y which is also induces an invertible map on the corresponding topologies. A basic task is to decide whether two spaces are equivalent in this sense or not. The surface of the coffee cup for example is equivalent in this sense to the surface of a doughnut but it is not equivalent to the surface of a sphere.

Many properties of manifolds can be understood by replacing them with **graphs**. A graph is a finite collection of vertices V together with a finite set of edges E, where each edge connects two points in V. The set V of cities in the US where the edges are pairs of cities connected by a street for example is a graph. The **Königsberg bridge problem** was definitely a trigger for the study of graph theory. Other examples of graphs are **polyhedra** the study of which is closely related to the analysis of surfaces. The reason is that one can see polyhedra as discrete versions of surfaces. In computer graphics for example, surfaces are rendered as finite graphs, using triangularizations. The **Euler characteristic** of a convex polyhedron is a remarkable topological invariant. It is

$$V - E + F = 2 \,,$$

where V is the number of vertices, E the number of edges and F the number of **faces**. This number is equal to 2 for connected polyhedra in which every closed loop can be pulled together to a point. This formula for the Euler characteristic is also called **Euler's gem**, a fact with an interesting history. **René Descartes** seems have stumbled upon it and written it down in a secret notebook. It was Leonard Euler in 1752 was the first to prove it for convex polyhedra. A convex polyhedron is called a **platonic solid**, if all vertices are on the unit sphere, all edges have the same length and all faces are congruent polygons. A theorem of Theaetetus states that there are only 5 platonic solids: [Proof: Assume the faces are regular n-gons and r of them meet at each vertex. Beside the Euler relation $V + E + F = 2$, a polyhedron also satisfies the relations $nF = 2E$ and $rV = 2E$ which are obvious from counting vertices or edges in different ways. This gives $2E/r - E + 2E/n = 2$ or $1/n + 1/r = 1/E + 1/2$. From $n \geq 3$ and $r \geq 3$ we see that it is impossible that both r and n are larger than 3. There are therefore only two possibilities: $n = 3$ or $r = 3$. In the case $n = 3$ we have $r = 3, 4, 5$ in the case $r = 3$ we have $n = 3, 4, 5$. We get five possibilities $(3,3), (3,4), (3,5), (4,3), (5,3)$ for forming pairs (n,r).] The pairs (n,r) which appeared in this proof are the **Schläfly symbol** of the polyhedron:

Name	V	E	F	V-E+F	Schläfli
tetrahedron	4	6	4	2	$\{3,3\}$
hexahedron	8	12	6	2	$\{4,3\}$
octahedron	6	12	8	2	$\{3,4\}$

Name	V	E	F	V-E+F	Schläfli
dodecahedron	20	30	12	2	$\{5,3\}$
icosahedron	12	30	20	2	$\{3,5\}$

The Greeks proved the fact more geometrically: Euclid showed in his "Elements" that at each vertex, we can attach 3,4 or 5 equilateral triangles, 3 squares or 3 regular pentagons. (6 triangles, 4 squares or 4 pentagons would lead to a too large total angle because each corner must have at least 3 different edges). **Simon Antoine-Jean L'Huilier** refined in 1813 Euler's formula to situations with holes: $V - E + F = 2 - 2g$, where g is the number of holes. For a doughnut with one hole we have $V - E + F = 0$. . Cauchy first proved that there are exactly 4 non-convex regular **Kepler-Poinsot** polyhedra. Their Euler characteristic can be different.

Name	V	E	F	V-E+F	Schläfli
small stellated dodecahedron	12	30	12	-6	$\{5/2,5\}$
great dodecahedron	12	30	12	-6	$\{5,5/2\}$
great stellated dodecahedron	20	30	12	2	$\{5/2,3\}$
great icosahedron	12	30	20	2	$\{3,5/2\}$

If two different face types are allowed but each vertex still look the same, one obtains 13 **semi-regular polyhedra.** They were first studied by **Archimedes** in 287 BC. Since his work is lost, **Johannes Kepler** is considered the first person since antiquity to describe the whole set of thirteen in his "Harmonices Mundi". The Euler characteristic $\chi = 2 - 2g$ is also useful for surfaces. It completely characterizes smooth compact surfaces if they are orientable. A non-orientable surface, the **Klein bottle** can be obtained by gluing ends of the Möbius strip. Classifying higher dimensional manifolds is more difficult and finding more invariants is part of modern research. Higher analogues of polyhedra are called **polytopes**. The **regular polytopes** are the analogue of the platonic solids in higher dimensions. Here they are for the first few dimensions:

dimension	name	Schläfli symbols
2:	Regular polygons	$\{3\}, \{4\}, \{5\}, ...$
3:	Platonic solids	$\{3,3\}, \{3,4\}, \{3,5\}, \{4,3\}, \{5,3\}$
4:	Regular 4D polytopes	$\{3,3,3\}, \{4,3,3\}, \{3,3,4\}, \{3,4,3\}, \{5,3,3\}, \{3,3,5\}$
≥ 5:	Regular polytopes	$\{3,3,3,\ldots,3\}, \{4,3,3,\ldots,3\}, \{3,3,3,\ldots,3,4\}$

Ludwig Schläfly found in 1852 that there are exactly six convex regular convex 4-polytopes or **polychora**. The expression "choros" is Greek for "space". Schlaefli's polyhedral formula tells that for any **convex polytope** in four dimensions, the relation $V - E + F - C = 0$ holds, where C is the number of 3-dimensional **chambers**. In dimensions 5 and higher, there are only 3 types of polytopes: the higher dimensional analogues of the tetrahedron, octahedron and the cube. A general formula $\sum_{i=0}^{d-1}(-1)^i V_i = 1 - (-1)^d$ gives the Euler characteristic of a convex polytop in d dimensions with i-dimensional parts V_i. The right hand side is 0 in even dimensions and 2 in odd.