

## Lecture 3: Geometry

Geometry is the science of **shape, size and symmetry**. While arithmetic dealt with numerical structures, geometry deals with metric structures. Geometry is one of the oldest mathematical disciplines and early geometry has relations with arithmetics: we have seen that the implementation of a commutative multiplication on the natural numbers is rooted from an interpretation of  $n \times m$  as an area of a **shape** that is invariant under rotational **symmetry**. Number systems built upon the natural numbers inherit this. Identities like the **Pythagorean triples**  $3^2 + 4^2 = 5^2$  were interpreted geometrically. The **right angle** is the most "symmetric" angle apart from 0. Symmetry manifests itself in quantities which are **invariant**. Invariants are one of the most central aspects of geometry. Felix Klein's **Erlanger program** program uses symmetry to classify geometries depending on how large the symmetries of the shapes are. In this hour, we look at a few results which can all be stated in terms of invariants. In the presentation as well as the worksheet part of this lecture, we will work us through 4 smaller miracles, **special points in triangles** as well as 4 gems, the theorems of **Pythagoras, Thales, Hippocrates** and **Feuerbach**. All of these examples illustrate the importance of the concept of symmetry.

Much of geometry is based on our ability to measure **length**, the **distance** between two points. A modern way to measure distance is to determine how long light needs to get from one point to the other. This **geodesic distance** generalizes to curved spaces like the sphere and is also a practical way to measure distances, for example with lasers. It bypasses the problem to determine first the underlying nature of the space in which we do geometry. Having a distance  $d(A, B)$  between any two points  $A, B$ , we can look at the next more complicated object, which is a set  $A, B, C$  of 3 points, a **triangle**. Given an arbitrary triangle  $ABC$ , are there relations between the 3 possible distances  $a = d(B, C), b = d(A, C), c = d(A, B)$ ? If we fix the scale by  $c = 1$ , then  $a + b \geq 1, a + 1 \geq b, b + 1 \geq a$ . For any pair of  $(a, b)$  in this region, there is a triangle. After an identification, we get the **moduli space**, an abstract space, which represent all triangles uniquely up to similarity. We will look at this in the presentation part and a worksheet if time permits.

A **sphere** is the set of points which have distance 1 from a given point. In the plane, the sphere is called a **circle**. A natural problem is to find the circumference  $L = 2\pi$  of a unit circle, the area  $A = \pi$  of a unit disc, the area  $F = 4\pi$  of a unit sphere and the volume  $V = 4 = \pi/3$  of a unit sphere. Measuring the length of segments on the circle leads to new concepts like **angle** or **curvature**. Because the circumference of the unit circle in the plane is  $L = 2\pi$ , angle questions are tied to  $\pi$ . The most **symmetric situation** of two lines crossing is when all 4 angles which appear are the same. This leads to the **right angle**.

Also **volumes** were among the first quantities, Mathematicians wanted to measure and compute. For example, a problem on **Moscow papyrus** dating back to 1850 BC explains the general formula  $h(a^2 + ab + b^2)/3$  for a truncated pyramid with base length  $a$ , roof length  $b$  and height  $h$ . An other great moment of mathematics is the determination of the **volume of the sphere** by Archimedes. Place a cone inside a cylinder. The complement of the cone inside the cylinder has on each height  $h$  the area  $\pi - \pi h^2$ . The half sphere cut at height  $h$  is a disc of radius  $(1 - h^2)$  which has area  $\pi(1 - h^2)$  too. Since the slices at each height have the same area, the volume must be the same. The complement of the cone inside the cylinder has volume  $\pi - \pi/3 = 2\pi/3$ , which is indeed half of the volume of the sphere.

The first geometric playground was **planimetry**, the geometry in the flat two dimensional space. Highlights are **Pythagoras theorem, Thales theorem, Hypochrates theorem**, and **Pappus theorem**, which we explore in a worksheet. Discoveries in planimetry are still made today. We see also a 19'th century 20th century discovery on the work sheet, the Feuerbach theorem. Greek Mathematics is closely related to history. It starts with **Thales** goes over Euclid's era at 300 BC, and ends with the threefold destruction of Alexandria 47 BC by the Romans, 392 by the Christians and 640 by the Muslims. Geometry was also a place, where the **axiomatic method** was brought to mathematics: theorems are proved from a few statements which are called axioms. The most famous are the 5 axioms of Euclid:

1. Any two distinct points  $A, B$  determines a line through  $A$  and  $B$ .
2. A line segment  $[A, B]$  can be extended to a straight line containing the segment.
3. A line segment  $[A, B]$  determines a circle containing  $B$  and center  $A$ .
4. All right angles are congruent.
5. If lines  $L, M$  intersect with a third so that inner angles add up to  $< \pi$ , then  $L, M$  intersect.

**Euclid** wondered whether the fifth postulate can be derived from the first 4. He called theorems derived from the first four the "absolute geometry". Only much later, with **Karl-Friedrich Gauss** and **Janos Bolyai** and **Nicolai Lobachevsky** in the 19'th century in **Hyperbolic space** the 5'th axiom does not hold. Indeed, geometry can be generalized to non-flat, or even much more abstract situations. Basic examples are geometry on a sphere leading to **spherical geometry** or geometry on the Poincare disc, a **hyperbolic space**. Both of these geometries are non-Euclidean. **Riemannian geometry**, which is essential for **general relativity theory** generalizes both concepts to a great extent. An example is the geometry on an arbitrary surface. Curvatures of such spaces can be computed by measuring length alone, which is how long light needs to go from one point to the next.

An important moment in mathematics was the **merge of geometry with algebra**. This giant step is often attributed to **René Descartes**. Together with algebra, the subject leads to algebraic geometry. We will see in this lecture also how algebra allows to automatize proofs.

Here are some examples of geometries which are determined from the amount of symmetry which is allowed:

Euclidean geometry	Properties invariant under a group of rotations and translations
Affine geometry	Properties invariant under a group of affine transformations
Projective geometry	Properties invariant under a group of projective transformations
Spherical geometry	Properties invariant under a group of rotations
Conformal geometry	Properties invariant under angle preserving transformations
Hyperbolic geometry	Properties invariant under a group of Möbius transformations

Since time and space on this 2 page summary is up, lets just water our mouths with pictures about the 4 special points in a triangle and with which we will begin. We want to see first, why in each of these cases 3 lines intersect in a common point. It is a manifestation of a **symmetry** present on the space of all triangles. Some **size** is constant, if we move on the space of all triangular **shapes**. It's Geometry!

