

# GOD NUMBERS FOR GRAPHS, GAMES AND GROUPS

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ABSTRACT. We describe and axiomatize finite solitaire puzzles and zero sum sequential games graph theoretically. Zermelo's theorem telling that there is a win for one of the players or a draw follows from the definitions. The god number is a geometric quantity that quantifies the number of moves necessary to solve the puzzle. In the solitaire case, the god number is the minimal distance from the initial state  $v$  to the solution space  $A$ . If  $v$  and  $A$  are not specified, the god number is the graph diameter. God number computations are related to combinatorial sorting problems and is a NP-complete problem in general even when restricted to concrete sliding problems. In the two-player case, the god number is a minimax critical value: it minimizes the maximal game event length over the set of all strategies. A strategy is a sub-graph of the game graph that contains the initial vertex. The definition is done so that a "mate in k" chess problem has god number k. As for examples: in the solitaire case, we look at group games like Rubik type problems, transposition games related to sorting, at sliding puzzles like the 15 puzzle or rainbow ball, or the tower of Hanoi. For two-player games, we illustrate the story using examples of small chess games, a small card game or tic-tac-toe type problems.

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1. IN A NUTSHELL

**1.1.** A **solitaire** is a finite simple directed graph  $G = (V, E)$  with an initial condition  $v \in V$  and a **solution set**  $A \subset V$ , for which the puzzle is considered solved. We use the term “puzzle” as a synonym to “one-player game” or “solitaire”. The game is not continued when the **equilibrium set**  $S$  and especially the **solution set**  $A \subset S$  is reached. The **equilibrium set**  $S$  is defined as the set of vertices in  $G$  with only incoming edges. A graph is **simple** if no self loops  $v \rightarrow v$ , nor multiple connections appear. It is possible to see an edge  $(a, b)$  as **undirected**, if both connections  $a \rightarrow b, b \rightarrow a$  are present. The **god number of a winnable solitaire** is the minimal path length from  $v$  to  $A$ . For undirected graphs, it is the **geodesic graph distance** between  $v$  and  $A$ . If a solitaire is non-winnable, it is custom to define the god number to be infinite. We can still then still look at maximal game events. Also, if  $S$  and so  $A$  are empty, we could look at the **maximal length of a game event**. Finding this is related to the question whether the graph is **Hamiltonian**.

**1.2.** Here is the graph theoretical axiom system for a two-player game. It will be outlined again in separate sections and summarized in a slightly more formal way at the end: a **two-player game** is a **bipartite directed graph**  $G = (V \cup W, E)$  with an **initial position**  $v$  and a set  $A$  of **win positions** in the set of equilibria  $S$ . The set  $V$  consists of all positions of the first player, while  $W$  is the set of positions of the second player. We refer to  $G$  as the **game graph**. The first player moves along edges pointing from  $V$  to  $W$ , the second payer moves on edges pointing from  $W$  to  $V$ . The set  $S$  of **end position** consists of all vertices without outgoing edges. The complement of  $A$  in  $S$  are **draw positions**. These are vertices representing game positions where the game can not be continued but where none of the players has won. A **game event** is a maximal simple path  $\gamma$  in  $G$  that starts at  $v$ . It could be a path of length 0 like if  $v \in S$ . That  $\gamma$  is **simple** means that it does not cover any vertex twice. **Maximal** means that it is not part of a strictly larger game event. A game event terminating in  $A$  is called **winning**, otherwise it is a **draw game event**: it can either end up in  $S$  or then trap itself, as it needs to be **self-avoiding**. The first player who reaches  $A$  is a winner. There are two scenarios for a draw: the draw event  $\gamma$  can either end in  $S \setminus A$  or then end in a vertex, where a continuation is not possible without hitting an already covered position and so violating the simplicity of  $\gamma$ .

**1.3.** A **strategy** is a sub-graph  $H$  of  $G$  that contains  $v$ .<sup>1</sup> A **win strategy for V** is a strategy  $H$  that does not remove any edges starting from  $W$  and which has the property that every game event in  $H$  ends in  $A \cap W$ . A **win strategy for W** is a strategy  $H$  that does not remove any edges starting from  $V$  such that game event in  $H$  ends in  $A \cap V$ . A **draw strategy** is a strategy that is neither a win strategy for  $V$ , nor a win strategy for  $W$ . In a draw strategy  $H$ , both players have limited their move options (removed edges) so that the win set  $A$  can not be reached in  $H$  by neither of the players.

**1.4.** In this frame work, **Zermelo’s theorem** for a game  $(G = (V, W, E), v, A)$  logically follows from the definition. Let us prove it nevertheless, as traditionally one uses backward induction on the finite game tree [49].<sup>2</sup> We also have **game graph**, not a **game tree** so that that argument anyway would not work in the graph theoretical frame work that we consider here.<sup>3</sup>

**Theorem 1** (Zermelo’s theorem). *There is either a win strategy for  $V$ , or a win strategy for  $W$  or a draw strategy.*

<sup>1</sup>We discuss this definition more below. Simplicity is intended.

<sup>2</sup>And because an AI pupil to which we have taught the axioms once did first not believe that it is as simple.

<sup>3</sup>Most games have game graphs, not game trees. There can be various ways to reach the same position. Indeed this happens in most games.

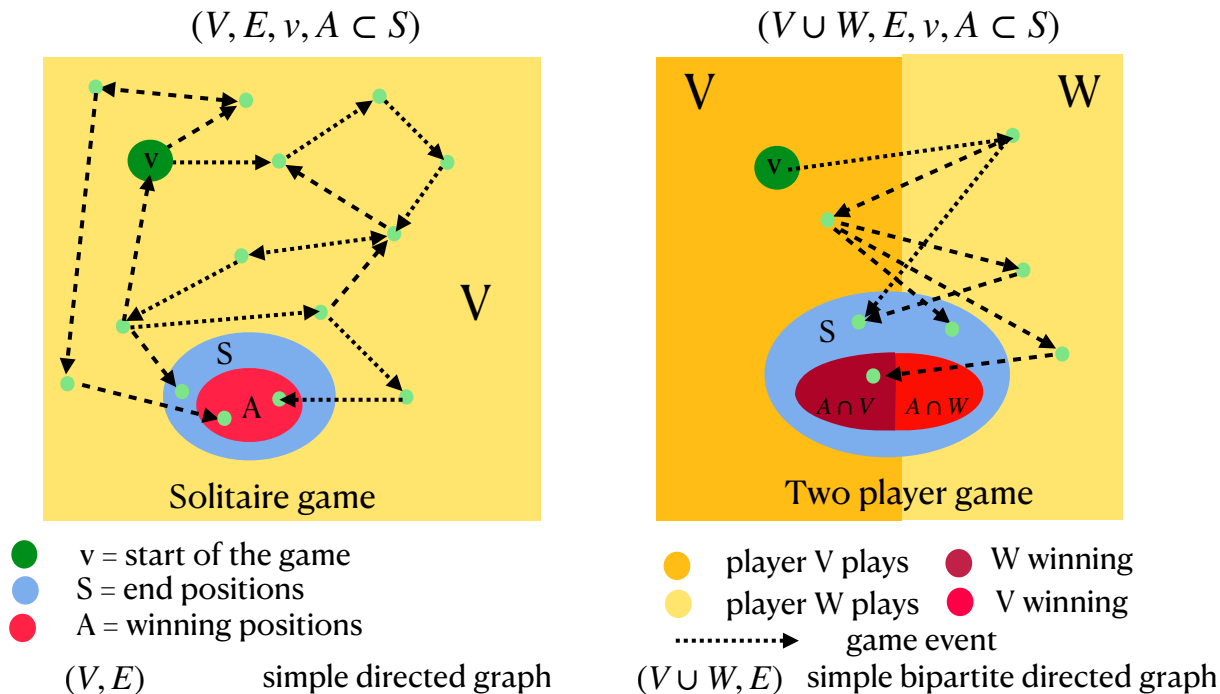


FIGURE 1. Both solitaires and two-player games are defined graph theoretically. In the solitaire case, the puzzle is either solvable or not. In the two-player case, there is a **Zermelo trichotomy**: a game is either a win for V (V can remove edges leaving V so that every game event lands in  $A \cap W$ ), a win for W (W can remove edges leaving W so that every game event lands in  $A \cap V$ ), or a draw. If both players can remove edges from their side so that no game event ends in A, we have a **draw**.

*Proof.* As  $G$  is a finite graph and game events are maximal simple paths in  $G$ , there are only finitely many game events that are possible. To each event, there is a definite value "win" or "loss" for either player. There are 4 logical possibilities. The possibility that is a win strategy for both can not occur: indeed, either  $V$  can limit the options so that no game event from  $W$  to  $A$  exists, or  $W$  can limit their options so that no game event from  $V$  to  $A$  exists. Or then both can construct together a strategy  $H$ , where  $V$  is allowed to trim outgoing edges from  $V$  and  $W$  is allowed to trim edges outgoing from  $W$  so that  $H$  has no game event ending in  $A$ .  $\square$

**1.5.** Zermelo's theorem have been described in [81]. It needed adaptations and corrections like [38]. It is discussed in [70]. A proof of Zermelo's theorem using induction is in [49] (Theorem 1.5). We will later remark that one could still use trees, but then we would have to look at the **self-avoiding walk tree** of the game graph.

**1.6.** In general, there is no procedure to find a win strategy in general, except by exhaustive case study [7]. Indeed, the task of finding the god number is in general a **NP complete** problem: the solitaire game in which one has to find a maximal complete sub-graph in a graph or the problem to find a Hamiltonian path in a graph are fundamental NP complete problems [39]. A concrete case of a solitaire that is NP complete is the  $n \times n$  generalization of the 15-puzzle. This class of games are discussed later on.

**1.7.** The **god number** for a given strategy  $H$  is the maximal length  $|\gamma|$  that a game event  $\gamma$  in  $H$  can have. The length  $|\gamma|$  of  $\gamma$  is assumed to be the number of edges in  $\gamma$  emanating from  $V$ . It is the number of play moves of player  $V$ . The length of the path as a "measure of loss" and is

defined in such a way that *if white moves and mates in n* means that  $V=$ "White" and that the god number is n. For a problem *black moves and mates in n* means  $V=$ "Black" and that the god number is n. The assumption  $v \in V$  assumes that  $V$  is the **first player**. The situation where the "second player" starts, can easily be realized by renaming.

**1.8.** To place this paper into the given mathematical landscape, we remark that our approach is elementary, geometric and entirely **graph theoretic**. It could be formalized in [19], as we only use notions from graph theory: directed graphs, subgraphs and paths. Other notions exist of course. There has been in particular other set-ups for "strategy". The book [49] for example defines a "strategy" a rule which tells in each situation where to go. There are probabilistic set-ups also, where a strategy is a probability measure on the possible moves. We mention some history in a separate section and especially comment that traditional game theory developed in the 1950ies by students of Albert Tucker focused on various forms of cooperation, which is necessary when looking at multi-player games. The difficulty of cooperation and collusion through the formation of coalitions already starts with three players.

**1.9.** All graphs are simple (no multiple connections, no loops) and unweighted (no additional properties are attached to edges or vertices). The graph data  $(V, E, v, A)$  or the bipartite directed graph data  $(V \cup W, E, v, A \subset S)$  determine the game entirely. The scope covers all deterministic sequential zero-sum 2-person games. It is especially part of **combinatorial games** [7, 2]. There is little relation with "graphical games" [40]. For solitaire games, the god number problem is a **combinatorial sorting problem**. Many game graphs are Cayley graphs of groups so that it is also part of **combinatorial group theory**. Puzzles and games deliver an accessible gateway to mathematics. Together with history as a gateway, it is a nice selling point. Recent example books to this would be [52, 57].

**1.10.** The setup could naturally be generalized to  **$n$ -player games**: take a finite simple directed  **$n$ -partite graph**  $G = (V = V_0 \cup \dots \cup V_{n-1}, E)$ , where edges  $(a \rightarrow b) \in E$  have the property that  $a \in V_k$  implies  $b \in V_{k+1 \bmod n}$ . Players play cyclically. The **initial position**  $v \in V_0$  and the **winning set**  $A \subset S$  are defined as before. The **winning set**  $A$  is assumed to be a subset of  $S = \{x \in V, \text{outdeg}_G(x) = 0\}$ . A **game event** is a maximal simple path  $(v_0, \dots, v_m)$  in  $G$  that starts at  $v_0 = v$ . A  $V_i$  **strategy** is a sub-graph  $H$  of  $G$  that contains  $v$  such that only edges emanating from  $V_i$  are removed. Formally,  $v \in V(H), E(G) \setminus E(H) \subset E_i$ , where  $E_i = \{(a, b) \in E, a \in V_i\}$ . A **game event** in  $H$  is a maximal simple path in  $H$  that starts at  $v$ . A non-empty proper subset of players  $I$  is called a proper **coalition**.<sup>4</sup> A strategy is called a  $V_I$ -strategy, if  $v \in V_I, E(G) \setminus E(H) \subset \bigcup_{i \in I} E_i$ . The **length**  $|\gamma|$  of a game event  $\gamma = (v = v_0, v_1, \dots, v_m)$  is the number of  $V_0$  edges in  $\gamma$ . A game event represents a **win for player**  $V_k$  if the final edge  $(v_{m-1}, v_m)$  in the path  $\gamma$  satisfies  $v_{m-1} \in V_k$  and  $v_m \in A_k = A \cap V_{k+1 \bmod n}$ . Coalition  $I$  is winning if it has a strategy, forcing every game event into  $A_I = \bigcup_{i \in I} A_i = \bigcup_{i \in I} (A \cap V_{(i+1) \bmod n})$ . It is **minimally winning** if no proper subset  $J \subset I$  is winning. An  $n$ -player game is a **draw**, if no coalition has a winning strategy For  $n$ -player games, the Zermelo trichotomy needs to be replaced with a study of **winning coalitions**. By definition, every game is either a draw or then winnable for at least one coalition. The set of all strategies is richer because **collusion** is possible: it can happen for a 3-player game already that the game is winnable for any coalition of size 2 but that is not winnable for any single player. Two players can band together to make sure that the third will lose. Since players can help each other in 3 or more player setups, we stick here to two- player games. For a winning coalition  $I$ , the god number is  $\text{God}_I(G, v) = \min_{H \in \mathcal{H}_I^{\text{win}}} \max_{\gamma \in \mathcal{E}(H)} |\gamma|$ , where  $\mathcal{H}_I^{\text{win}}$  is the set of winning I-strategies and  $\mathcal{E}(H)$  the set of game events in  $H$ .

<sup>4</sup>We leave away "proper" to avoid a pleonasm: the set of all players is not a coalition in colloquial language.

**1.11.** We taught Chat GPT these axioms and asked it to construct a simple example of a **3 player game**, where only coalitions of two players can win. Here is the example the machine came up with. We could only think of the **truel** in the movie *The good, the bad and the ugly*. So lets call this the “truel” example. It has  $\mathbf{God}(G, v) = 1$  for every winning coalition of size 2. This takes into account that only the number of moves of the player  $V_0$  are counted.

**1.12.** The following paragraph is a machine generated result. We slightly edited the answer of the machine. Even so our set up is non-standard, the machine understood it well and creatively generate an example to the prompt: *”Please construct a small game example to the attached axiom system, where collusion occurs, meaning that there is no win for a single player but a coalition of two can force a win for them”*. The machine was fed the above paragraph about multi-player games and also had previously been taught the axioms in the special case of one player or two player games, to have more context.

**1.13. Example:** (“truel”) <sup>5</sup> *As defined, players  $V_0, V_1, V_2$  move cyclically  $V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_0$ . The game is defined by the finite directed 3-partite graph with  $V_0 = \{v, t_2\}, V_1 = \{p_1, p_2\}, V_2 = \{q, t_1\}$  and directed edges  $v \rightarrow p_1, v \rightarrow p_2, p_1 \rightarrow t_1, p_2 \rightarrow q, q \rightarrow t_2$ . The terminal equilibrium set is  $S = \{t_1, t_2\}$ . We pick  $A = S$  (which is allowed). The game graph is visualized in the Figure below. Under the normal-play convention, a terminal vertex in  $V_{k+1}$  is a win for the player  $V_k$  who made the last move. Hence  $t_1 \in V_2$  is a win for  $V_1$ , because the last move is  $p_1 \rightarrow t_1$  made by  $V_1$ . Similarly,  $t_2 \in V_0$  is a win for  $V_2$ , because the last move is  $q \rightarrow t_2$ , made by  $V_2$ . Thus the individual terminal win sets are  $A_0 = \emptyset, A_1 = \{t_1\}, A_2 = \{t_2\}$ . a) We first check the coalitions. (i) First, the coalition  $I = \{V_0, V_1\}$  can force a win. Player  $V_0$  restricts the first move to  $v \rightarrow p_1$ . Then player  $V_1$  moves  $p_1 \rightarrow t_1$ . The game ends at  $t_1$ , which is a win for  $V_1$ , hence a win for the coalition  $\{V_0, V_1\}$ . (ii) Second, the coalition  $I = \{V_0, V_2\}$  can force a win. Player  $V_0$  restricts the first move to  $v \rightarrow p_2$ . Then the move  $p_2 \rightarrow q$  is forced, and player  $V_2$  moves  $q \rightarrow t_2$ . The game ends at  $t_2$ , which is a win for  $V_2$ , hence a win for the coalition  $\{V_0, V_2\}$ . (iii) Third, the coalition  $\{V_1, V_2\}$  can force a win. This coalition cannot restrict the first move of  $V_0$ , so both possibilities  $v \rightarrow p_1$  and  $v \rightarrow p_2$  remain. But this is harmless. If the play goes through  $p_1$ , then  $p_1 \rightarrow t_1$  ends the game at a win for  $V_1$ . If the play goes through  $p_2$ , then  $p_2 \rightarrow q \rightarrow t_2$  ends the game at a win for  $V_2$ . Therefore every possible game event ends in a win for one of the coalition members. Hence  $\{V_1, V_2\}$  is also a winning coalition. b) Now, we check that no single player can force a win. The player  $V_0$  has no terminal win position in  $S$ , since  $A_0 = A \cap S = \emptyset$ . Thus  $V_0$  cannot force a win. The player  $V_1$  cannot restrict the first move of  $V_0$ . If  $V_0$  chooses  $v \rightarrow p_2$  then the play continues  $p_2 \rightarrow q \rightarrow t_2$  which ends in a win for  $V_2$ , not for  $V_1$ . Hence  $V_1$  cannot force a win. Similarly, the player  $V_2$  cannot restrict the first move of  $V_0$ . If  $V_0$  chooses  $v \rightarrow p_1$  then the play continues  $p_1 \rightarrow t_1$  which ends in a win for  $V_1$ , not for  $V_2$ . Hence  $V_2$  cannot force a win. We have now proven that this is a simple 3-player game in which no single player can win, but where coalitions of 2 players can win.*

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<sup>5</sup>Generated by GPT 5.4, prompted on May 8, 2026. It was manually slightly formatted and shortened.

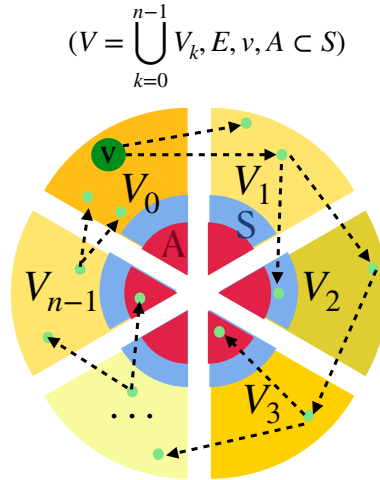


FIGURE 2. For a multi-player game with more than 2 players, the Zermelo analog of coalitions is already more complicated. Coalitions can form.

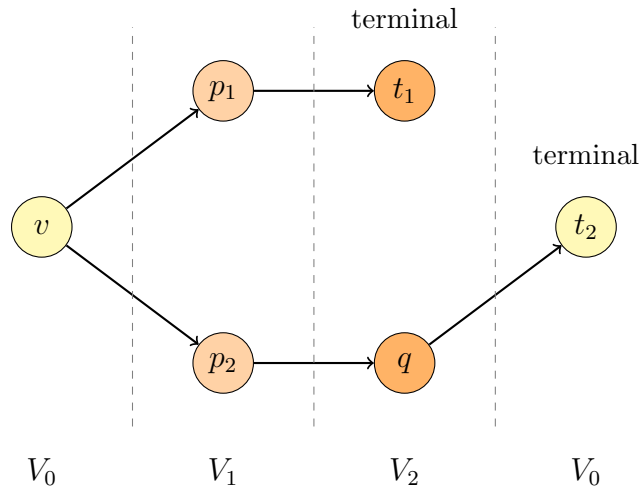


FIGURE 3. A simple 3 player game, where only coalitions of 2 players can win. This example (as well as the graph) were machine generated. The machine was only fed the above simple axiom system, without giving any additional help. We wanted a simple example.



FIGURE 4. An exhibit of a few toys and games, we have played with when working on this project.

## 2. INTRODUCTION

**2.1. God numbers** quantify optimal solutions for **deterministic, finite, zero sum, sequential** games. The term “god number” has appeared first in the context of the **Rubik cube puzzle** [74, 69, 37], where the number is now known to be 20 in the half turn metric [67]. The term was popularized in the early 1980ies by “cubists” and mathematician **David Singmaster**, who distributed notes on the “magic cube” in 1979 already. (They were published in 1981 [74]). For a two-player game, the **god number** is a **minimax optimization problem** for the game graph. The prototype is a **chess problem** “mate in  $n$ ” has god number  $n$ .

**2.2.** A problem “*draw in  $n$* ” for example tells that for some strategy, the maximal path length in it leads to a draw. In a game with god number  $n$ , the analysis of a game only needs to be done in a **neighborhood graph**  $B_n(v)$  of  $v$ .<sup>6</sup> “Draw” either forces a game event to reach  $S \setminus A$  or then ends in a dead end, not being able to continue without crossing itself. “Draws” can also be a “theoretical draws” meaning that they are undisputed end positions. In chess for example, if we reach a position where both players only have a king and a horse. The game does not need to be continued. We could bake this into the geometry by removing any outgoing edges at such points of the game graph. As more advanced players are, as more they can prune the game graph and also label positions  $w$  as points in  $A$  which are clearly winning. Pruning means to remove any outgoing edges from  $w$ .

**2.3.** The class of games that can be described as such is close to [81], who would rather see it as a playground within set theory rather than geometry. Zermelo’s article was only three and a half pages long and needed various clarifications [70]. Restricting to graph situations covers most puzzles and games. The geometric language of graphs allows to use intuitive geometric notions.

**2.4.** The geometric setup models the usual assumptions:

- The term **zero sum** means that if one players wins the other playes loses. It is not possible that both win. If both lose, the game is a draw.
- **Deterministic** means that both players at any time have the same full access to information, that no external randomness nor third party input can happen.
- **Sequential** means that players always take alternate turns. This is encoded in the bipartite structure of the graph. It also is an ingredient to deterministic.
- **Finite** means that the entire game has only finitely man states. The game is so encoded as a **finite simple graph**, where the nodes are the possible configurations and where the oriented edges are the possible moves.
- **Finite events** are forced by the assumption that all **game events** describing game events must be maximal simple paths starting at  $v$ . The case for having no self intersections is done such that there are only finitely many game events.
- **No cycles** is encoded in the simplicity of the game events. It is an important assumptions we want a finite set of game events. The convention in chess, limiting the number of repetitions can easily be bent to this. If a position appears three times, just cut away the loops which have led to it and stop the game after the first repeat.

**2.5.** The standard  $3 \times 3 \times 3$  **Rubik cube**  $G$  is is a popular solitaire game. [74, 69, 37]. Its god number  $\text{God}(G)$  is known to be 20. [67]. As for all Cayley group games, it is the diameter of the Cayley graph of a finite simple graph with  $R = 8!12!(3^7)(2^{10})$  elements. Since this is larger than  $2^{65}$ , we can not hope to draw out the entire game graph.

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<sup>6</sup>This is a ball of radius  $n$  in the graph distance of the directed graph. Note that  $w \in B_1(v)$  does not imply necessarily that  $v \in B_1(w)$ .

**2.6.** In **chess**, the number of nodes of the graph has an upper bound  $10^{47}$ . Claude Shannon estimated the number of game positions as  $64!/(32!8!2!^6) \sim 10^{43}$  without promotion [72]. The best upper bound appears to be  $2 * 10^{40}$  [75]. It is not the size but the complexity of this game which often makes it impossible to compute the god number of chess yet. To illustrate this, look at the  $5 \times 5 \times 5$  Rubik cube (which is called "professor cube"). This game has  $24!^3/24^{12}$  positions, more than the number of chess positions.

**2.7.** The size of the neighborhood graphs  $B_r(v)$  of the initial position  $v$  in chess grows fast and is known as the OEIS number A048987. After 1 play of white we have 20 positions, after a play of white and black, there are 400. The current status is after 15 half moves. On smaller chess boards or with initial conditions with very few figures (chess puzzles), we can draw out all the positions.

**2.8.** One reason to restrict to 2 players is that for more than two-players, like a **3 player chess**, the possibility of **collusion** is possible already. With more players, one also has to clarify the order in which players play like taking turns in a circular way. A geometric description is not as simple as for 2-player games because Zermelo's theorem does not generalize in an obvious way. It is not true that either one of the player wins or that we have a draw. We can have a perfectly symmetric case with 3 players where one of the players needs to pick a strategy favoring one of its opponents and so decide the winner. The game itself does not determine that as in the case of 2-player setup.

**2.9.** Also the sequential assumption is important. If the players could move simultaneously, like in a **rock-paper-scissors** game or **prisoner dilemma** game, the two players would have both to move at the same time. Since in that case only partial information is available, **Nash equilibria** conditions start to matter. If randomness was allowed, it would force to include a probability space that to generate the input. This would force us to look at very large graphs or then look at conditional expectations.

**2.10.** We can exploit that many popular puzzles and games naturally are given as finite directed graphs (even so the graphs are too large to be drawn out like in chess). This allows for a rather intuitive axiomatic setup which is simpler, if one compares it with the existing alternatives. The geometric frame work only appears to be a limitation if one would insist that one can construct the entire game graph. But such limitations appear everywhere: we can not explore the graph of all integers  $\mathbb{N}$ , where two are connected if one is a factor of the other for example. This graph exists but we can only see a small part of it. Similarly, if games are small enough, we have a reasonable chance to oversee the entire graph and measure the god number.

**2.11.** For the foundations of game theory, see [53]. The geometric setup restricts to games that have entered **popular culture**. Remarkably, the number of board games or solitaire games that can be described using graphs is quite rich.

**2.12.** This paper is a collaborative author effort as part of a reading class Math/CS 91r, which met weekly in the spring 2026 semester at Harvard. Initially, we split things up as follows among the different group members but there was some overlap. This is reflected also in the current document as we did not optimize the document for brevity.

- "transposition games" (15-game, color shift, graph, peg)
- "group type games", (rubik, clock, hungarian rings)
- "pile games" (nim, tower of hanoi, mankala )
- "chess type games" (chess, checkers, go, morris)
- "card type games", (solitaire, card ordering)

- "geometric games" (tic-tac-toe, connect 4, Hex)

**2.13.** As for originality, one should note that graph theoretical approaches to games are hardly original, this probably goes back to Cayley already. But it should be pointed out again that traditionally one looks at "game trees". The closest in literature is the recent book [23], where a strategy (or **policy**) is formally seen a function from all paths to edges (see page 16 in the open preprint version of [23]). The language uses **arenas** or "game structure" which could be interpreted as a game graph. There are other major differences in that **quantitative** or **qualitative** winning conditions are considered. There are other major differences in that all vertices have an outgoing edge in [23]. We feel that this is very restrictive. Already simple games like peg solitaires or Tic-Tac-Toe two player games have positions without outgoing edge. One could mend it by assuming that such positions have self-loops.

**2.14.** Zermelo's work of 1913 can be seen as the beginning of formal **game-tree search**. But Zermelo did not use the language of graphs. This should probably be attributed to Kalmar from his 1928/29 work on abstract games [38]. Kalmar's article was written in German would use terminology like "**taktik**" instead of strategy. Kalmar's work considerably clarifies Zermelo's theorem. See [20].

**2.15.** The elementary graph-theoretic formalism presented here is close to finite games with perfect information and turn-based multiplayer games on graphs which are also called "arenas" with reachability objectives. We think that a novelty of the present formulation is the **elementary subgraph interpretation** of strategies and seeing the associated "god numbers" as measuring the worst-case number of moves by the initial player needed under a winning strategy. The simplicity of the set-up is maybe best illustrated in that the theorem of Zermelo is immediate and that many commonly known puzzles and games can be covered with the language.

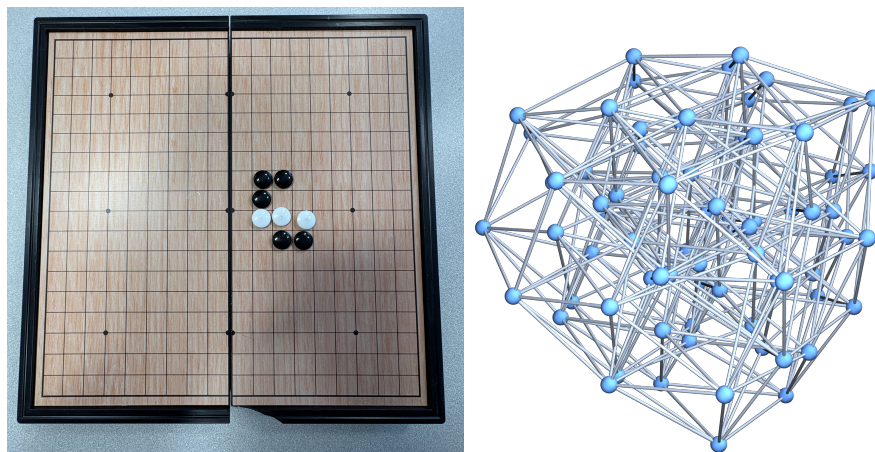


FIGURE 5. **Infinite connect 5** (Gomoko) can also be played on finite boards like on a  $19 \times 19$  **go game board**. "Go" itself has been played for more than 4000 years. Go type games or hex type games are examples of **topological games**, where the winning positions are geometrically defined.  $5 \times 5$  go was determined to be a win for the first player (Black) [22], like for "**microgo**" on  $2 \times 2$  board which requires only to look at  $3^4 = 81$  possible positions. The game graph of microgo is seen to the right.

**2.16.** In the case of group puzzles, the concept of “god number” seems first have emerged when Rubik type puzzles came up in the late 1970ies. It definitely appears in the write-up of [74] (page 34) in the form of “**God’s algorithm**”. Mathematicians certainly have contemplated about the minimal number of moves to solve a puzzle before. For 2-player games like chess, the word “god number” is not used, but most “solve a position in n-moves” belong to god number problems. We say more about the history in a separate section.

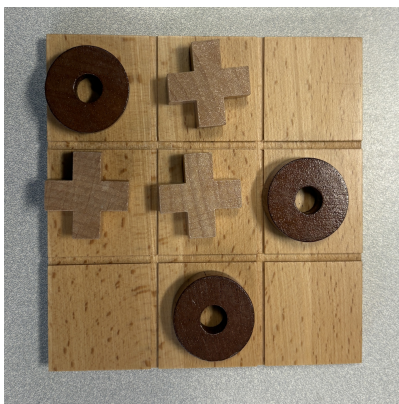


FIGURE 6. Tic Tac Toe is infamous due to its simplicity and its multi-cultural value, appearing in blockbuster Hollywood movies like “WarGames”. It is a 2-player game that is well known to lead to a draw. We can easily map out the entire game graph.

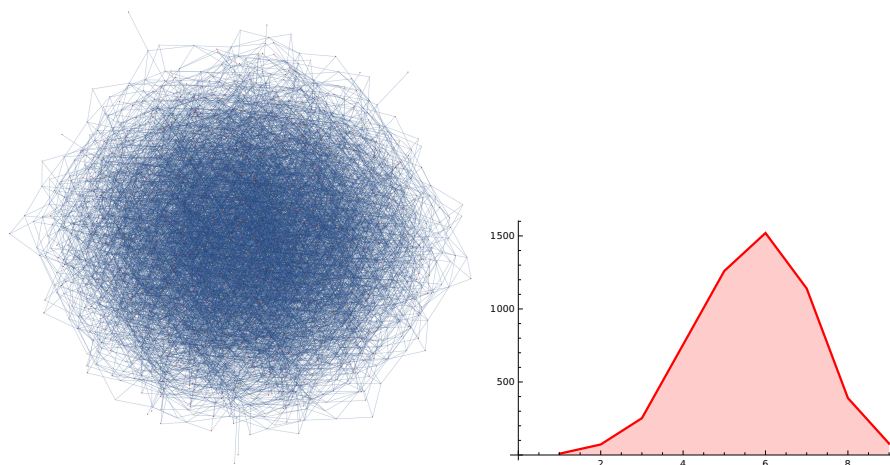


FIGURE 7. The game graph of the standard Tic Tac Toe game has 5478 vertices. This is the number of eligible the  $3^9$  possible  $3 \times 3$  matrices with entries in  $\{0, 1, 2\}$  (we need to look a subset because the number of white stones is always one larger or equal than the number of black stones). The graph ball sizes  $B_r(0)$  starting from the initial  $v = 0$  position are 1, 10, 82, 334, 1090, 2350, 3870, 5010, 5400, 5478. The difference sequence is plotted to the right. It is known as the **BFS layer profile**. The **ball** (=neighborhood graph)  $B_9(0)$  covers the game graph. The maximal game length is 5 (if measured in the number of moves of White).

**2.17.** For a  $4 \times 4$  Tic-Tac-Toe game, we already have to weed through  $3^{16} = 43'046'721$  positions. Exploring the full graph is already tough with standard computer algebra.

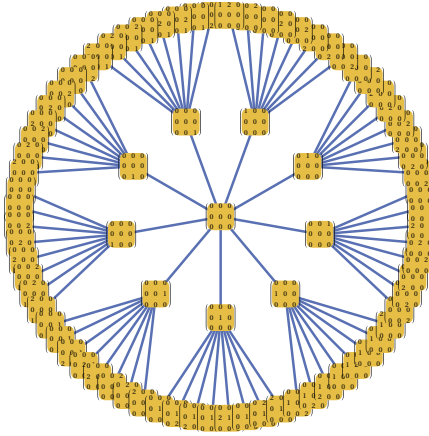


FIGURE 8. We see the ball  $B_1(0)$  of play radius 1 centered at 0. It consists of 82 possible Tic Tac Toe games in which both players have played one stone. We count the number of plays of the first player, as in chess.

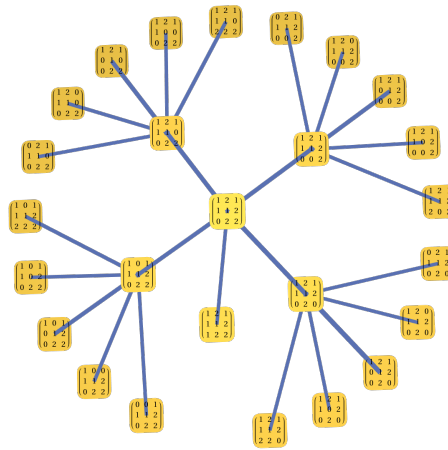


FIGURE 9. We see the ball  $B_1(0)$  of play radius 1 centered at a winning position for the first player.

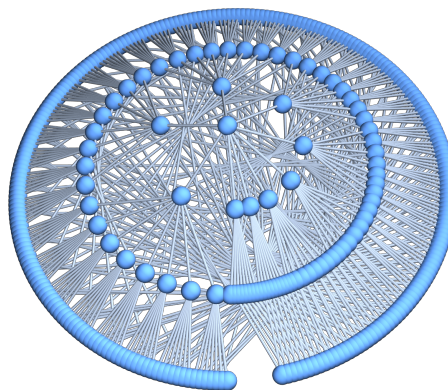


FIGURE 10. We see the graph after an other move by the first player.

**2.18.** Here is an example of a Cayley graph for which the BFS profile is **not unimodal**.<sup>7</sup> The graph  $G$  is defined by the generators  $48, 88, 138, -48, -88, -138$  in the Abelian group  $(\mathbb{Z}_{713}, +)$ . Its BFS layer profile  $(1, 6, 18, 38, 66, 102, 140, 138, 140, 60, 4)$  has a small dip  $140, 138, 140$ .

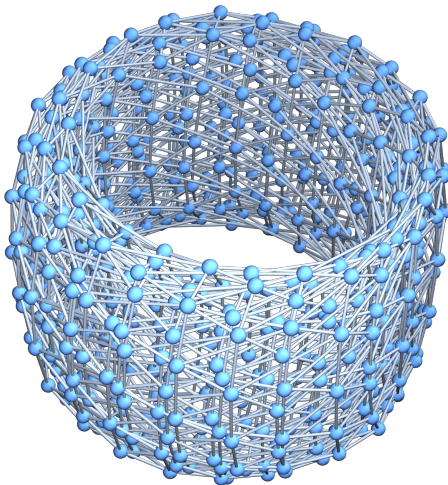


FIGURE 11. We see the Cayley graph of an Abelian group in which 6 generators  $a, b, c, a^{-1}, b^{-1}, c^{-1}$  are given. Its diameter is 10. The BFS layer profile has a concave up segment. Most Cayley graphs show a unimodal BFS layer profile.

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<sup>7</sup>This example was found AI assisted.

### 3. SOLITAIRE GAMES

**3.1.** A **solitaire game** or **puzzle** is a finite simple directed graph  $G = (V, E)$ , a vertex  $v \in V$  as **starting position** and a subset  $A \subset V$  representing the **solved positions**. Solved positions also also called **win positions**. The player starts at a position  $v \in V$ . The goal is to find the shortest game event from  $v$  to  $A$  if it exists. The puzzle moves do not need to be reversible. A “peg solitaire” is an example, where one can not turn back. That game can be played on many geometries, the most common being sub-graphs of lattices.

**3.2.** The **god number**  $\text{God}(G, v, A)$  of a winnable puzzle is the **geodesic graph distance**  $\min_{w \in A} d(v, w)$ . If  $v$  and  $A$  are not described, the god number  $\Gamma(G)$  of the graph can still be defined as the **diameter of the directed graph**. When talking about the god number of a solitaire game, we usually mean the **game graph diameter**. This is the solution of a **maximin variational problem**

$$\text{God}(G, A) = \max_{v \in V} \min_{\gamma} |\gamma| .$$

If  $A$  is not specified, the god number of the game graph  $(V, E)$  can be given as

$$\text{God}(G) = \max_{v \in V, w \in V} \min_{\gamma: v \rightarrow w} |\gamma| ,$$

where  $\gamma$  is a simple path from  $v \in V$  to  $w \in W$ . This is the obviously the same than the **graph diameter**, the maximal geodesic distance between any two vertices in  $V$ . Also the diameter is a maximin problem  $\max_{v \in V, w \in V} \min_{\gamma: a \rightarrow b} |\gamma|$ .

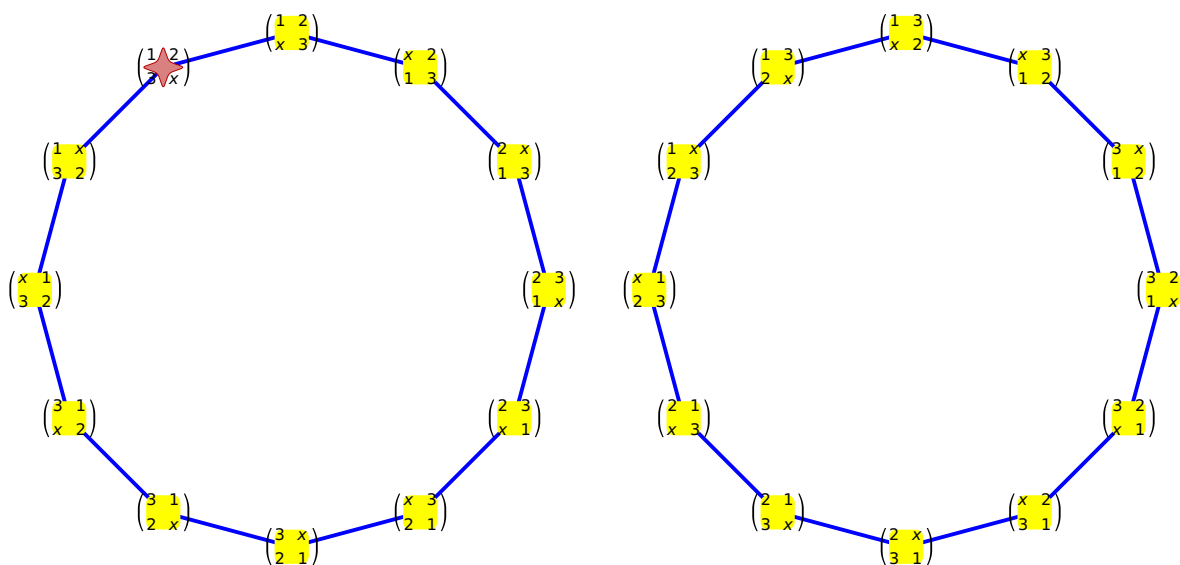


FIGURE 12. The **3-puzzle** game  $G$  is an example of a sliding game. It is the  $2 \times 2$  analog of the **15 game** which is played on a  $4 \times 4$  board. The graph has two connected components; each is a cyclic graph of order 12. The god number in the winnable component is  $\text{God}(G) = 6$ . For the 8-puzzle, the god number is 31. For the 15 game, the god number is 80. For the  $n \times n$  generalization, its computation is NP complete.

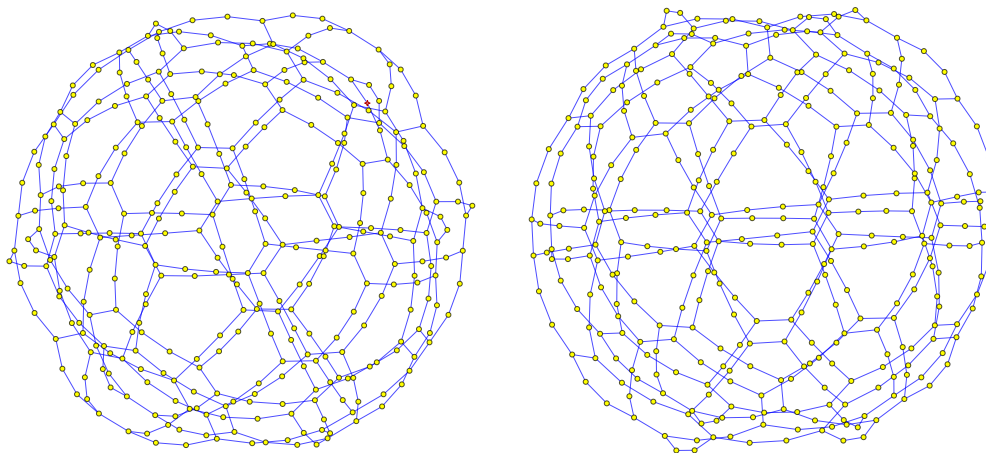


FIGURE 13. The 5 puzzle game is the  $2 \times 3$  analog of the 15 game. The graph has two connected components. The god number is 21.

**3.3.** In the code section we have included the code which generated the following picture. We show here only the first connected component, which represent the solvable cases: the position

$$\begin{bmatrix} 4 & 5 & x \\ 1 & 2 & 3 \end{bmatrix} \text{ has distance 21 from } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & x \end{bmatrix}.$$

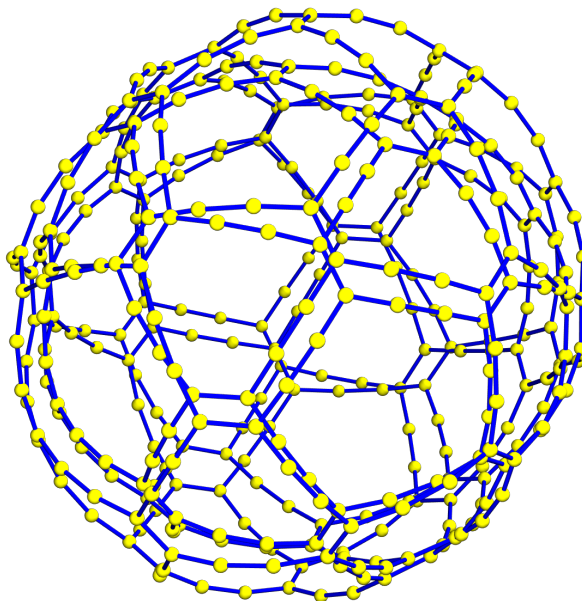


FIGURE 14. The solvable connected component of the previous picture is here seen embedded in 3D. The game graph has  $360 = 6!/2$  vertices and 420 edges. All vertex degrees are equal to 2 or 3.

**3.4.** Whether a solitaire has a win strategy depends on whether  $v$  is in the same connected component than one of the points  $A$ . If  $v$  is disconnected from  $A$ , there is no solution. There are various mechanisms which can render  $G$  disconnected. The most obvious one is an invariant non-constant quantity.

**3.5. Examples:**

a) For the **15-puzzle**, the initial position is given by a permutation in  $S_{16}$ . The graph has two connected components. The parity of the permutation and the signature of the  $l_1$ -distance of the hole to  $(4, 4)$  is an invariant.

b) For a finitely presented permutation group with generators that all have even signature, the signature of the initial condition is an invariant, the reason being that the composition of two even permutations is even.

c) In the Rubik cube case, the total rotation of all cube corners has to be a multiple of  $2\pi$ . A situation, where one cube is turned by 120 degrees has been called a **"quark"** configuration in [29]. Three different quarks model a **Baryon**, two different quarks where one is turned by 120, the other by 240 is a **Meson**. We see so caricatures of **Hadron's** in finite groups. While quarks can not be realized, Hadron combinations can.

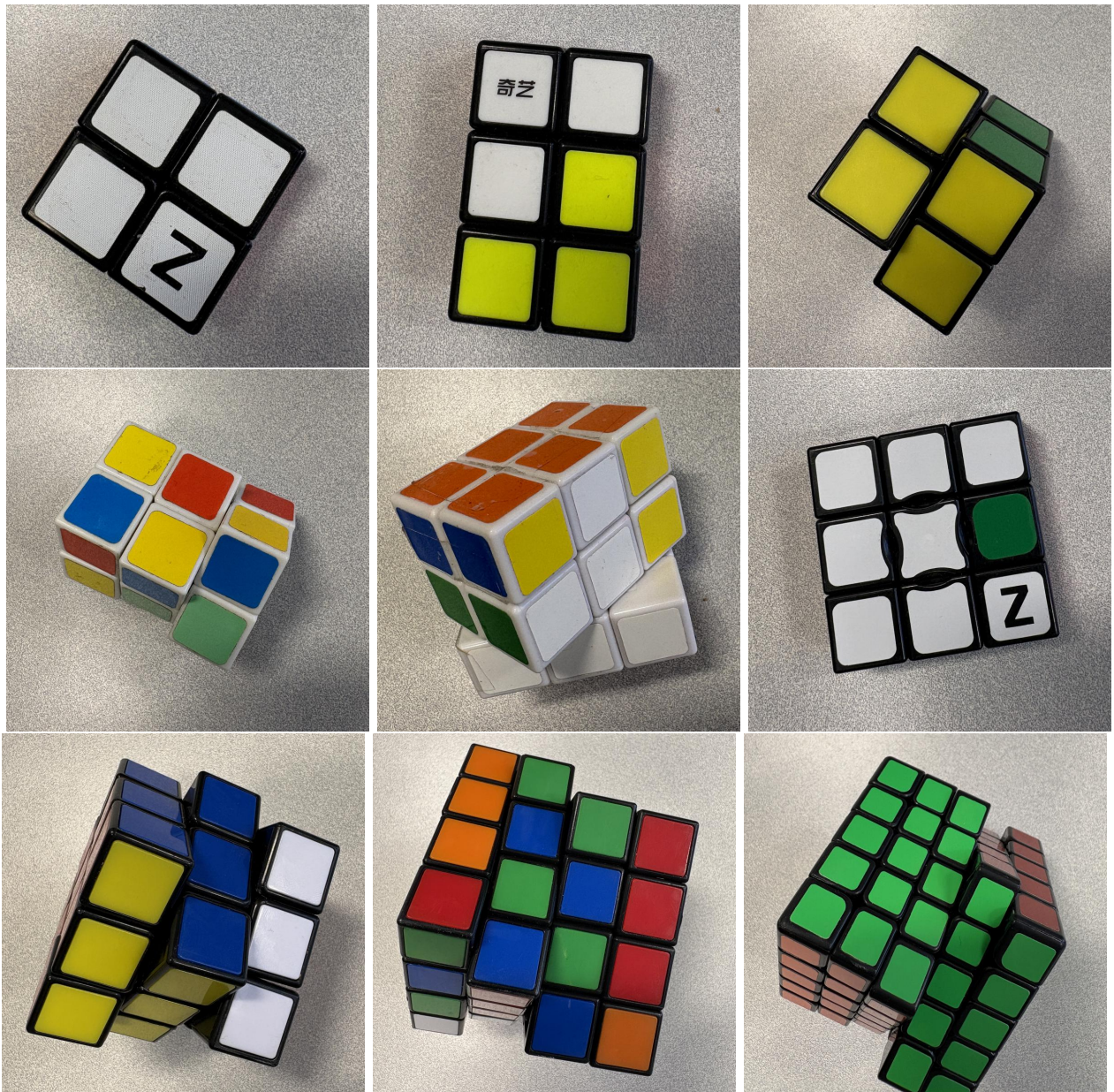


FIGURE 15. Rubik puzzles of size  $2 \times 2 \times 1$ ,  $2 \times 3 \times 1$ ,  $2 \times 2 \times 2$  (pocket cube),  $2 \times 2 \times 3$ ,  $2 \times 3 \times 3$ ,  $3 \times 3 \times 1$ ,  $3 \times 3 \times 3$ ,  $4 \times 4 \times 4$  and  $5 \times 5 \times 5$  (professor's cube).

## 4. TWO PLAYER GAMES

**4.1.** A **two-player game** is a finite simple directed bipartite graph  $G = (V \cup W, E)$  together with a vertex  $v$  giving the starting position and a subset  $A$  of **win positions** in  $S$ , where  $S$  is the set **end positions**, the set  $S$  of vertices without an emanating edge. Points in  $S$  are game positions, where no move is possible any more. The set  $S$  of end positions is part of the rules of the game and sometimes artificially added. For example, it can be a position where naturally no move is possible any more. It can be a position which is defined to be a defeat like if the **king is mate**, or then a position which is declared to be hopeless, like a position where both players do not have enough resources to win and where it would be pointless to continue. An example in chess would be if both players only have a king and a horse, in which case only a “help mate” could render a win.

**4.2.** An **game event** is a maximal simple path  $\gamma$  in  $G$  that starts from  $v$ . The assumption is that on each directed edge, the path needs to follow the direction of the digraph. Simple means that no vertex is covered twice. Pawn moves in chess are examples of directed moves. Capturing an opponents piece is an other example of a non-reversible move in chess. The **length**  $|\gamma|$  **of**  $\gamma$  is the number of  $V$ -vertices in  $\gamma$ . We choose this and not the length of  $\gamma$  in order to have “mate in  $n$  moves” be a problem with god number  $n$ . We also designate a subset  $A \subset S$  as the set of **win positions**. The complement of  $A$  in  $S$  are declared to be **draw positions**.

**4.3.** A **strategy** is a sub-graph  $H$  of  $G$ . A **strategy for  $V$**  is a strategy that does not remove any edges starting from  $W$ . A **strategy for  $W$**  is a strategy that does not remove any edges starting from  $V$ . A **win strategy for  $V$**  is a strategy for  $V$  such that every game event ends in  $A \cap W$ . A **win strategy for  $W$**  is a strategy for  $W$  such that every game event ends in  $A \cap V$ . A **draw strategy** is a strategy that is neither a win strategy for  $V$  nor for  $W$ .

**4.4.** The **god number of a strategy  $H$**  is the maximal possible length of a game event in  $H$ . For a game that can be won by  $V$ , the **god number** is  $\text{God}(G) = \min_H \max_{\gamma=\gamma(H)} |\gamma|$ , where  $H$  runs over all  $V$ -strategies leading to a win of  $V$ . The other cases are similar. While minimax principles for two-player zero-sum games go back John von Neumann, the min-max situation in our case is much simpler. The length of a game event is the potential loss. To get the god number, we minimize the maximal loss over all possible game strategies.

**4.5. Examples:**

a) Let us look at a chess game on a  $2 \times 4$  board, where both parties only have king or queen and no other figures are present. This is obviously a draw case. The set of configurations with only kings present is declared to be part of  $S$  as the game does not continue there. The initial player  $V = (\text{White})$  has a winning strategy. It removes all initial plays except capturing the other queen.  $W$  can now use the empty  $W$ -strategy because it is forced to recapture the queen. But then we are in  $S$ .

b) Lets look at a chess game on a  $4 \times 3$  board again with only king and queen on both sides at the corners. Now the initial player  $V = (\text{White})$  has an obvious win strategy: capture the queen. It is mate in 1. The god number is 1.

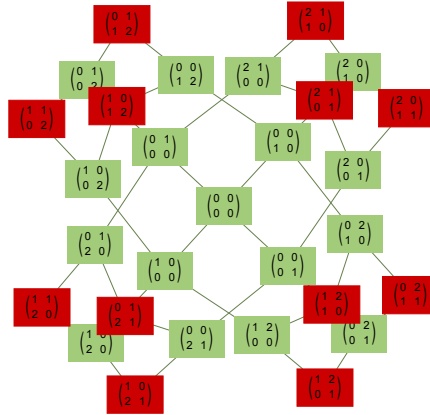


FIGURE 16. we see the simple  $2 \times 2$  tic-tac-toe game has no draw. The initial player  $V$  always wins. The god number is 2. The initial player always wins in 2 moves. The empty board is the initial position  $v$ . The vertex set  $V$  is the set of boards with an even number of stones. The vertex set  $W$  is the set of boards with an odd number of stones.  $S$  is the set of boards where all squares are occupied or where a row, a column or a diagonal have the same color.

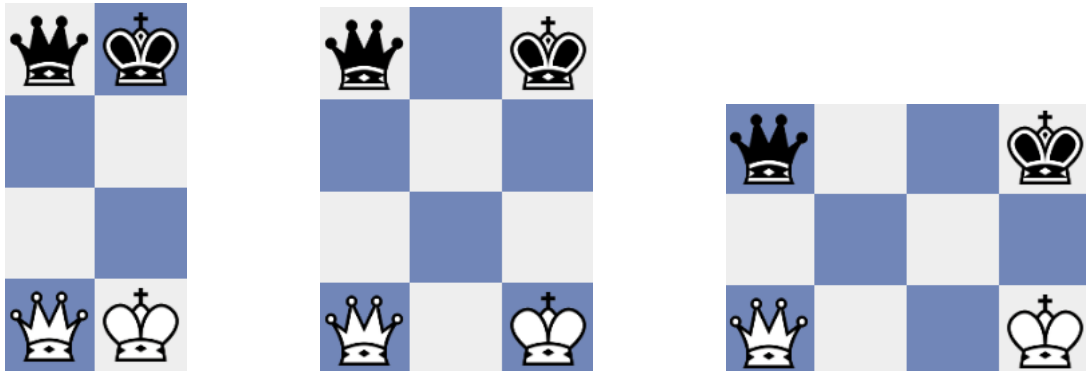


FIGURE 17. In the left case, we have a draw. The god number is 1 because after both parties have moved, we only have kings left, which is part of  $S$ . In the middle case, there 8 possible legal moves initially, one of them (catch the queen) leads to a white win, two to black wins and the rest to draws. A game engine tells that the god number is 11 to get to a mate. In the right case, there is an obvious mate in 1. The god number is 1.

## 5. TRANSPOSITION GAMES

**5.1.** In this section we describe a class of solitaire games that come from **finitely presented groups** that are generated by **involutions**, and so are **Coxeter group**. In this case it is in general the symmetric group. We are not interested so much in the group but in the **Cayley graph** of the group generated by a set of transpositions. Finding its diameter and so the god number gives an answer to a **topological sorting problem**.

**5.2. T.** he set of transpositions that generate the subgroup  $G$  are defined by the **transposition graph**  $\Gamma$  which is a finite graph on the vertex set  $V = \{1, \dots, n\}$ , the edges being the generators. he problem is to find the god number of the game graph defined by a connected graph  $\Gamma$  with vertex set  $V$ . The reason to restrict to connected graphs is that the god number of of the disjoint union  $\Gamma = \Gamma_1 \cup \Gamma_2$  of two graphs is the sum of the god numbers of the components.

Rather than using the language of **presentations**, where the group elements are words, we use the **permutation representation** and work in  $S_n$ , the permutation group of  $n$  elements.

**5.3.** It is easy to see by induction that the **transposition graph**  $\Gamma$  is connected if and only if the generated group is  $S_n$  and that if  $\Gamma$  is the disjoint union of  $m$  graphs  $\Gamma_k$ ,  $k = 1, \dots, m$  with vertex cardinality  $n_k$ , then the group  $G$  is the Cartesian product  $\Gamma_{n_1} \times \dots \times \Gamma_{n_m}$ . This is known as the "**connected transposition graph criterion**". From the introduction to [26] p 2483: *the folklore fact that  $\Gamma$  is connected is equivalent to  $G = S_n$* . See also [47]. Since we did not find the statement explicitly proven anywhere (it might be too simple to deserve this), here is the proof.

**Theorem 2** (Folklore result on transposition game graphs). *Let  $\Gamma = (V, E)$  be a finite simple connected graph with  $n = |V|$  vertices. The edges  $(a, b) \in E$  define transpositions in the permutation group  $S_n$ . The subgroup of  $S_n$  generated by these transpositions is  $S_n$ .*

*Proof.* If  $\Delta$  is a subgraph of  $\Gamma$ , it generates a subgroup as less generators are used. Let  $\Delta$  be a spanning tree in  $\Gamma$ . It still has  $n$  vertices, but now it has  $(n - 1)$  edges. We made the problem seemingly harder by restricting the permutations to a smaller set of generators and show that  $\Delta$  already generates  $S_n$ .

We use induction with respect to  $n$ . For  $n = 1$ , there is no edge and  $S_1$  is the trivial group. This is the induction foundation. As for the induction step, assume that the statement is true for all trees with  $n - 1$  vertices. Take a tree  $\Delta$  with  $n$  vertices and pick a leaf  $b$ , (a vertex with vertex degree 1). It defines a unique edge  $(a, b)$  and so a flip  $(a, b)$ . Since the tree  $\Delta \setminus b$  has  $n - 1$  vertices, its edges generate the group  $S_{n-1}$  by the induction assumption. Take an arbitrary permutation  $\pi \in S_n$ . (Case 1): If  $\pi_n = n$ , we can use permutations in  $\Delta \setminus b$  to order it to  $(1, 2, \dots, n)$ . (Case 2): If  $\pi_n < n$ , assume that  $n$  has label  $j < n$  so that it is  $\pi = (\pi_1, \dots, \pi_{n-1}, j)$ , where one of the labels  $\pi_j$  is  $n$ . Use permutations from edges in  $\Delta \setminus b$  to rearrange this to  $(1, \dots, n, j)$ . Now apply the new flip  $(a, b)$  allowed in the full tree  $\Delta$  to get to  $(1, \dots, j, n)$ . Again use permutations from edges in  $\Delta \setminus b$  to do the final adjustment reaching the solved state  $(1, 2, \dots, n - 1, n)$ .  $\square$

**5.4.** The Cayley graph  $G$  has a natural **geodesic metric**  $d(x, y)$  on the graph  $G$ . The god number is the diameter of  $G$  with respect to this metric. Note that in general, the Cayley graph is a directed graph. It is often assumed however that if  $a$  is a generator in a group, then also  $a^{-1}$  is a generator. It is important to specify this when looking at god numbers. The **half turn metric** on the Rubik cube for example, allows to turn any face by any angle 90, 180, 270. The **quarter turn metric** only allows to use turns by 90 or 270 degrees. There is also the **slice turn metric** in which one can turn any layer by any angle. The god number in the slice turn metric is either 18, 19 or 20. The God number in the quarter turn metric is 26. The God number in the most commonly used half turn metric is 20.

**5.5. Breadth-First Search** (BFS) generates the graph and then determine its god's number. The graph can be built up by building **neighborhood graphs** or ball  $B_r(v)$  of radius  $r$ . In a Cayley graph, this can be used to find the diameter of the Cayley graph, the distance between a two points  $(a, b)$  is the same than the distance to the points  $ca, cb$ , where  $c$  is an other group element. To talk in the language of **differential geometry**, Cayley graphs are **objects of constant diameter**, similarly as curves of constant width like the filled in **Reuleux triangle**, which is an object for which the diameter can be computed from any point.

**5.6.** We can define an other metric on  $G$  which is in general different but allows to estimate the god number without generating the entire graph. This distance depends on the graph  $\Gamma$ : If  $M$  be the **distance matrix** of the connected  $\Gamma$ , define the **Gamma distance** on the symmetric group  $G$  as  $d_\Gamma(x, y) = \lfloor \sum_{k=1}^n M_{x(k), y(k)} / 2 \rfloor$ .

**5.7.**  $d_\Gamma$  is a metric: it is symmetric, commutative and satisfies the triangle inequality. It is also a lower bound for the geodesic metric  $d_\Gamma(x, y) \leq d(x, y)$ . But it is in general smaller. If  $\Gamma$  is the **octahedron** graph  $\Gamma = K_{2,2,2}$  for example, then the god number is 7, while the Hausdorff Gamma distance is 6. Even simpler is  $\Gamma = K_3$ , where the graph diameter of  $G$  is 2, while the  $d_\Gamma$  the Hausdorff distance is 1.

**5.8.** One of the simplest cases is the triangle  $K_3$  (complete graph of 3 elements), where we have 6 possibilities.  $\langle a = (1, 2), b = (2, 3), c = (1, 3) | a^2 = b^2 = c^2 = (ab)^3 = (ac)^3 = (bc)^2 \rangle$ . This puzzle can be solved in 3 or less cases. The hardest case is to translate  $v = (3, 2, 1)$  to  $a = (1, 2, 3) \in A$ . The shortest path is  $(3, 2, 1) \rightarrow (2, 3, 1) \rightarrow (2, 1, 3) \rightarrow (1, 2, 3)$ .

**5.9.** It is an interesting question to relate the topology of  $\Gamma$  with the god number. The diameter plays a role. We can for example ask, in which cases we can use **spanning tree** in  $\Gamma$  and get the same god number. An example we studied a bit more detailed is whether the star graph produces the same god number than the wheel graph. A bold question to ask is whether for large enough graphs  $\Gamma$ , it is always possible to find a spanning tree in  $\Gamma$  on which the god number is the same. This is of interest in logistic sorting problems as it is easier to organize a sorting mechanism along a spanning tree.

**5.10.** Here are some small examples. Denote by  $\lfloor x \rfloor$  the floor of  $x$ . It is the largest integer smaller or equal than  $x$ .

Graph $G$	God(G)
a) $K_n$ complete graph	$n - 1$ .
b) $P_n$ path graph	$n(n + 1)/2$ .
c) $C_n$ cycle graph	$\lfloor (n^2 + 1)/4 \rfloor$ [65]
d) $S_n$ star graph	$\lfloor 3n/2 \rfloor$ sketched in [1]
e) $W_n$ wheel graph	between $n$ and $\lfloor 3n/2 \rfloor$ .

**5.11.** Our proof of the circular c) appears to be slightly simpler than what can be seen in [65]. Together with [78] it produces the result in [65]. The paper [1] for the star graph gives an upper bound.

*Proof.* a) Induction. If an  $(n + 1)$ 'th point is added, we just need one more transposition.

b) This is bubble sort. The worst case is to sort the reversed string, as it needs  $n - 1 + n - 2 + n - 3 + \dots + 1 = n(n + 1)/2$  swaps.

c) This is the best studied case. The graph distance is  $m^2$  if  $n = 2m$  and  $m(m - 1)$  if  $n = 2m + 1$ . The reason is that we have to move distance  $m$  maximally for  $m$  vertices. A full proof is in [65]. See also [78] for the upper bound. As for the lower bound a concrete example is  $\pi(k) = k + \lfloor \frac{n}{2} \rfloor \pmod n$ , which has distance  $d(\pi, \iota) = \lfloor \frac{n^2+1}{4} \rfloor$  from the identity  $\iota(k) = k$ . Proof: If an edge  $e$  in  $\Gamma = C_n$  is removed, it gives a path graph  $P_n$ . Label its vertices  $\{1, \dots, n\}$ . For any permutation  $\sigma$ , define  $P_{e,\sigma}(x)$  to be the position of  $x$  in  $\{1, \dots, n\}$  that  $x$  occupies in the order determined by our cut  $e$ . Note that  $P_{e,\iota}(x) = x$ . Define

$$F_e(\sigma) := \sum_{x=1}^n |P_{e,\sigma}(x) - x|.$$

We have  $F_e(\iota) = 0$ . If  $d(\sigma', \sigma) = 1$  then  $|F_e(\sigma') - F_e(\sigma)| \leq 2$  because a transposition exchanges the positions of exactly two labels, say  $x$  and  $y$ . Therefore, if  $(d(\sigma, \iota) = t)$ , then  $F_e(\sigma) = |F_e(\sigma) - F_e(\text{id})| \leq 2t$ . It follows that  $d(\sigma, \iota) \geq \frac{F_e(\sigma)}{2}$ .

Case (i) In the case  $n = 2m$ , chose the edge  $e$  between  $m$  and  $m + 1$  so that  $P_{e,\pi}(x) = x - m$ . This gives  $F_e(\pi) = \sum_{x=1}^{2m} |P_{e,\pi}(x) - x| = \sum_{x=1}^{2m} m = 2m^2$  and so  $d(\pi_{2m}, \iota) \geq \frac{F_e(\pi)}{2} = m^2 = \lfloor \frac{(2m)^2+1}{4} \rfloor$ .

Case (ii) in the case  $n = 2m + 1$ , let  $e$  be the edge between  $m + 1$  and  $m + 2$ . The labels  $1, \dots, m$

are shifted to the right by  $m + 1$  steps, and the labels  $m + 1, \dots, 2m + 1$  are shifted to the left by  $m$  steps. We have  $F_e(\pi_{2m+1}) = \sum_{x=1}^m (m + 1) + \sum_{x=m+1}^{2m+1} m = m(m + 1) + (m + 1)m = 2m(m + 1)$  so that  $d(\pi_{2m+1}, \iota) \geq \frac{F_e(\pi_{2m+1})}{2} = m(m + 1) = \lfloor \frac{(2m+1)^2 + 1}{4} \rfloor$ .

In the circle, where we deal with the integer sequence A002620.

d) For the star graph, an upper bound is covered by the wheel case. The star graph appears in [1] (Theorem 7).

e) This is covered in two separate lemmas. It follows from d) as a wheel graph has more edges than a star graph. The lower bound follows because it is contained in the complete graph.  $\square$

**5.12.** The next two lemmas establish also part e) in the table. The star graph is a subgraph so that the god number known since [1] is larger or equal than in the wheel case. We see in experiments that the difference between star and wheel case is maximally 1 and that for larger  $n$ , the wheel and star graph have the same god number. (There is also a more detailed numerical table in the code section.) It is an interesting question, whether  $\text{God}(W_{n+1}) = \text{God}(S_{n+1})$  for large enough  $n$ . Of course  $\text{God}(W_{n+1}) \leq \text{God}(S_{n+1})$  because we can on the wheel graph also use swaps on the rim. It seems however that these **rim swaps** can be avoided for large  $n$  (this is an open question equivalent to the above question whether the god numbers of the star and wheel graph agree for larger  $n$ ).

**5.13.** We do not know an exact formula for the god number of  $W_n$ . A natural conjecture is that it is the same than for the star graph for larger  $n$ . A trivial estimate is  $\text{God}(W_{n+1}) \leq \text{God}(S_{n+1}) \leq 3(n-1)$  as three moves allow to switch any two points in the boundary keeping the center invariant. The additional "parking position" at the center renders  $O(n^2)$  complexity sorting on the circular graph (or path graph) to a linear  $O(n)$  complexity.

**Lemma 1** (Wheel upper bound).  $\text{God}(W_{n+1}) \leq \lfloor \frac{3n}{2} \rfloor$

*Proof.* Let  $c$  be the center of the wheel. Denote by  $r_1, \dots, r_n$  the **rim vertices** of  $W_{n+1}$ . Because  $W_{n+1}$  is connected, any permutation  $\pi \in S_{n+1}$  is a product of transpositions  $\pi_1 \cdots \pi_\ell$ . Then either  $\pi_i(c) \neq c$  (which means that we use transpositions on the star) or then  $\pi_i(c) = c$  (which means that we use transpositions on the rim). In the first case,  $\pi_i$  is of the form  $\pi_i = (c, a_1, a_2, \dots, a_k)$ , where each  $a_j$  is some  $r_i$ . We have the factorization

$$(c, a_1, a_2, \dots, a_k) = \underbrace{(c a_k)(c a_{k-1}) \cdots (c a_1)}_{k \text{ transpositions}} .$$

A cycle of length  $k + 1$  that contains  $c$  can be reached in  $k$  moves on the star. In the second case,  $\pi_i$  is of the form  $\pi_i = (a_1 a_2 \dots, a_k)$ , where all  $a_j$  are some  $r_i$ . Then we have the factorization

$$(a_1 a_2 \dots, a_k) = \underbrace{(c a_1)(c a_k)(c a_{k-1}) \cdots (c a_2)(c a_1)}_{k+1 \text{ transpositions}}$$

So a cycle of length  $k$  that does not contain  $c$  can be reached in  $k + 1$  moves on the star.

Define  $\Pi := \{\pi_i \mid 1 \leq i \leq \ell\}$  to be the set of our disjoint cycles. Then

$$\Pi = \underbrace{\{\pi_j \mid \pi_j(c) = c\}}_{:=A} \sqcup \underbrace{\{\pi_j \mid \pi_j(c) \neq c\}}_{:=B}$$

By disjointedness of the  $\pi_i$ ,  $|B| = 1$  or  $|B| = 0$ . Let  $|A| = \beta$  and let the unique  $\tau \in B$  (if at all) be  $\tau = (c a_1 \dots, a_m)$ . Let the cycles in  $A$  have lengths  $k_1, \dots, k_\beta$ . Since all the cycles are disjoint and  $\pi$  permutes at most  $n$  vertices, we have  $m + \sum_{i=1}^\beta k_i \leq n$ . By the above, we can achieve  $\pi$  in at most

$$m + \sum_{i=1}^\beta (k_i + 1) = m + \sum_{i=1}^\beta k_i + \beta \leq n + \beta$$

moves. Now observe that each  $\sigma \in A$  has length at least 2, so  $2\beta \leq \sum_{i=1}^{\beta} k_i \leq n \implies \beta \leq \lfloor \frac{n}{2} \rfloor$ . Thus we see that  $d(\pi, \text{id}) \leq n + \beta \leq n + \lfloor \frac{n}{2} \rfloor = \lfloor \frac{3n}{2} \rfloor$  if and only if  $\text{God}(W_{n+1}) \leq n + \lfloor \frac{n}{2} \rfloor$ .  $\square$

**5.14.** Now lets turn to the lower bound, where need to give a specific configuration.

**Lemma 2** (Wheel lower bound).  $\text{God}(W_{n+1}) \geq n$

*Proof.* Similarly as in the upper bound of the circular case, we give a permutation  $\pi \in S_{n+1}$  such that  $d(\pi, \iota) \geq n$ . For any  $\sigma \in S_{n+1}$ , we define  $P_{\sigma}(x)$  to be the vertex of  $W_{n+1}$  that  $x$  occupies. Define

$$D(\sigma) = \sum_{x \in \{c, r_1, \dots, r_n\}} |P_{\sigma}(x) - x| .$$

We observe that a single legal move swaps the labels on the endpoints of some edge, and each of them moves by 1. Thus their distance from home can decrease by at most 1, so  $D$  can decrease by at most 2 in a single move. Therefore, if we require a minimum of  $t$  moves to take  $\sigma$  to  $\iota$ , then  $0 = D(\iota) \geq D(\sigma) - 2t \implies t \geq \frac{D(\sigma)}{2}$ . For  $m = \lfloor \frac{n}{2} \rfloor$  define  $\pi \in S_{n+1}$  by  $\pi(c) = c$ ,  $\pi(r_i) = r_{i+m}$ , where we work  $\pmod n$ . For  $n \geq 4$ ,  $r_{i+m}$  is not adjacent to  $r_i$  on the rim, so  $d(r_i, r_{i+m}) = 2$ , (where we traverse the center for shortest distance). Since  $\pi(c) = c$ , we have

$$D(\pi) = \sum_{i=1}^n d(r_{i+m}, r_i) = \sum_{i=1}^n 2 = 2n .$$

$\square$

**5.15.** The question which game graphs of transposition games are Hamiltonian is related to the **Lovasz vertex-transitive graph conjecture**. We noticed experimentally that all game graphs from transposition games have the Hamiltonian graph property. For the octahedron  $\Gamma = K_{2,2,2}$  for example we have  $|V| = 720, |E| = 4320$ . It is a Hamiltonian graph. This is indeed a known result from 1975: [43]. We call it the **transposition game theorem**:

**Theorem 3.** *Every connected finite graph  $\Gamma$  defines a transposition game for which the game graph is Hamiltonian.*

**5.16.** We also looked at the size of the **last layer** of complete graph puzzles. These are the points furthest away from a point. For  $K_5$ , the last layer has exactly 3 configurations all of cycle type  $(2, 2, 1)$  and all with the hole at vertex 0. For  $K_6$ , the last layer has 35 configurations, split between cycle types  $(2, 2, 2)$  (with 15 representatives) and  $(3, 2, 1)$  (with 20 representatives), with hole positions distributed across all vertices. We have code producing the BFS layer profile for any graph.

**5.17.** Finally, lets look at a topological example. We take a discrete **Möbius strip**  $\Gamma_1$  and a discrete **Cylinder strip**  $\Gamma_1$ . They both have 8 vertices and 16 edges and 8 triangular faces; but the cylinder is oriented, while the Möbius strip is not. We can now look at the game graphs  $G_i$  of both. They are both finite graphs with  $8! = 40320$  vertices and 322560 edges. But the God numbers are different. For the Möbius strip, we have  $\text{God}(G_1) = 10$ . For the cylinder, we have  $\text{God}(G_2) = 11$ . In the code section we have both mathematica and GAP procedures what establish these numbers independently.

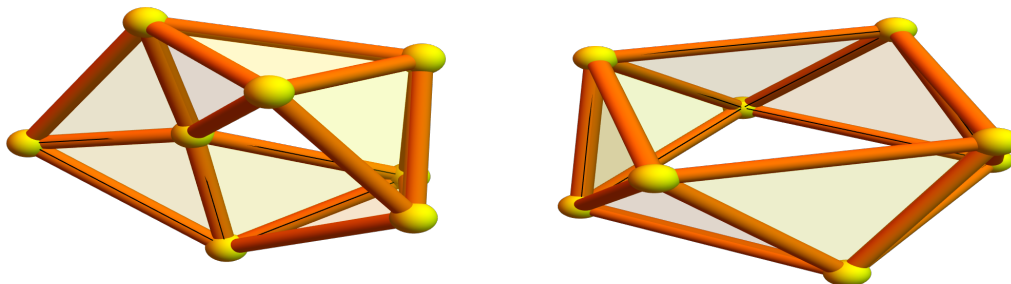


FIGURE 18. The Moebius strip and Cylinder both generate game graphs  $G_1, G_2$ . These two graphs are both Cayley graphs of  $S_8$  and both have the same constant vertex degree 16, and so the same number of edges. But they have different diameter 10 resp. 11.

## 6. SLIDING PUZZLES

**6.1. Sliding puzzles** generalize the classic 15-puzzle ( $\Gamma = C_4 \times C_4$ ) or the **magic ball**  $\Gamma$  is the **icosahedron graph**) to arbitrary connected graphs. Sliding puzzles for connected  $\Gamma$  produce game graphs  $G$  that are in general more interesting than the game graphs of transposition games, as the latter are always  $S_n$ . Wilson’s theorem [80] classifies the graphs.

**6.2.** Given a graph  $\Gamma = (V, E)$  on  $n$  vertices, one of the tokens is called a **hole**. A legal move slides the hole into an adjacent position, opening the hole where it had been. From a node  $v \in G$  only transpositions involving the hole can be used. The sorting is now localized near the hole. The set of all reachable permutations again produces a game graph  $G$  whose diameter is then the God number.

**6.3.** The history of the 15-puzzle (where  $\Gamma$  is the grid graph  $C_4 \times C_4$ ) is iconic and well documented: after a precursor was shown around by **Noyes Chapman** in 1874, he applied for a patent in 1880. A craze was driven by Sam Loyd’s offer of \$1000 for swapping tiles 14 and 15. Johnson and Story [36] demonstrated that a permutation parity is an invariant. implying that only half of the  $16!$  positions are reachable. Finding the god number of the general  $n \times n$  version of the puzzle is an NP complete problem [62].

**6.4.** Computational results for the  $(n^2-1)$ -puzzle proceeded with the development of **iterative deepening** in the context of **pattern databases**. [44] introduced **iterative-deepening**. [64] showed that the 8-puzzle’s god’s number is 31. [13] established that less than 81 moves are necessary for the 15-puzzle in 1999 via parallel BFS on ZRAM, extending Gasser’s earlier search that had found 9 states at depth 80. [46] confirmed and extended such exhaustive implicit-search computations using disk-based breadth-first search and stated 17 states at depth 80. [45] built the first optimal solver for individual 24-puzzle instances. The best bounds are  $152 \leq G \leq 205$ , the upper bound improved from 208 in 2016. [63] proved that finding a shortest solution to the  $(n^2-1)$ -puzzle is NP-hard. [59] gave a  $\Theta(n^3)$  real-time algorithm and showed  $\Theta(n^3)$  is also the growth rate of the God’s number itself. The graph-theoretic generalisation is due to [80]. A modern textbook approach is [5]. For a discussion of higher dimensional puzzles, see [11].

**6.5.** Here is an overview of the God’s number table for the  $(n^2-1)$ -puzzle with a single-tile metric:

Puzzle	Grid	# states $(n^2)!/2$	God's number	# hardest states	Reference
3-puzzle	$2 \times 2$	12	6	1 (antipode on 12-cycle)	trivial
8-puzzle	$3 \times 3$	181,440	31 (24 multi-tile)	—	[64]
15-puzzle	$4 \times 4$	$1.046 \times 10^{13}$	80 (43 multi-tile)	17	[13]
24-puzzle	$5 \times 5$	$7.76 \times 10^{24}$	$\in [152, 205]$	open	[45]
35-puzzle	$6 \times 6$	$\approx 1.9 \times 10^{41}$	unknown	—	—
$(n^2-1)$ -puzzle	$n \times n$	$(n^2)!/2$	$\Theta(n^3)$	—	[59]

**6.6.** There is also a **multi-tile metric**, where consecutive blank moves in the same direction count as one. For the 15-puzzle, an upper bound is 43, for the 8-puzzle it is 24 (OEIS A087725) gives the records 0, 6, 31, 80, ... for  $n = 1, 2, 3, 4$ . Finding some solution is polynomial in  $n$ : **Parberry's algorithm** uses  $\Theta(n^3)$  moves [59]. Finding a **shortest** solution is NP-hard [63]. It remains NP-hard to approximate within any additive constant, though a polynomial-time constant-factor approximation exists.

**6.7.** The game has a graph theoretical version which we discuss a bit more in an other section. Let  $\Gamma$  be a connected simple graph with  $n$  vertices in which one of them is called “blank” or “hole”. A legal move swaps the blank with an adjacent label. The resulting configuration graph  $G(\Gamma)$  has one vertex per labeling and an edge per move. The **eccentricity** of the solved vertex  $w$  in its connected component is defined to be the maximal shortest path distance from  $v$  to any other vertex.

**6.8.** The game graph  $G$  generated by these moves depends only on  $\Gamma$ . Wilson's theorem [80] characterizes it completely. It is the analog of the fact that for a connected graph the transposition game defined by a connected graph  $\Gamma$  on  $n$  vertices is the symmetric group  $G = S_n$ . Here is the statement of **Wilson's theorem on sliding graphs**:

**Theorem 4** (Wilson). *Let  $G$  be a finite simple 2-connected graph with  $n \geq 4$  vertices. The sliding-puzzle group  $G$  for  $\Gamma = C_n$  is  $\mathbb{Z}_n$ . For  $\Gamma = \theta_0$ , the group is  $\text{PGL}_2(\mathbb{F}_5)$  of order 120. If  $\Gamma$  is bipartite but not  $C_n$  nor  $\theta_0$ , it is the **alternating group**  $A_n$ . Otherwise, it is the full symmetric group  $S_n$ .*

**6.9.** For the 15 puzzle, because the  $4 \times 4$  grid  $\Gamma$  is bipartite. Wilson's result gives  $A_{15}$ , a group of order  $15!/2$ .

**6.10.** In a sliding game on a graph  $\Gamma$ , one of the vertices is marked and transpositions are only allowed if this special point is involved. The prototype is the 15 puzzle, where the underlying graph is the  $4 \times 4$  grid graph. The graph  $G$  of the game is in general not the Cayley graph of a group as the possible generators can depend on the position. In the 15 game, if the hole is at the boundary there are less legal moves than if the hole is inside.

**6.11.** For the **magic rainbow puzzle ball**, the graph  $\Gamma$  is the **icosahedral graph**. We have not been able to get the god number there with our general program as the game graph is too large For the magic ball analog where  $\Gamma$  is the octahedron graph however,

**6.12.** For the 8 and 11 puzzles, we also ran C programs. <sup>8</sup> The 8 puzzle game is the  $3 \times 3$  has god number 31. The 11 puzzle game is the  $3 \times 4$  analog. It has god number 53. This diameter 53 is known and OEIS A151944 [55] which credits [46]. Our Mathematica procedures to generate the small graphs are not feasible for the 15 puzzle any more. Storing the graph with  $16! = 20922789888000$  vertices would require dozens of terabytes of RAM.

<sup>8</sup>The C programs were obtained from asking ChatGPT to translate the Python implementation of the standard **Lehmer-code ranking** of permutations and breadth-first search on the implicitly generated sliding-puzzle graph.



FIGURE 19. Magic Rainbow puzzle ball.

**6.13.** Given a graph  $G$  and a vertex  $v$ , one can look at the **neighborhood graph**  $B_r(v)$  which consists of all vertices in distance  $r$  or less from  $v$ . This defines a **layer partition** or **spheres**  $S_r = B_r \setminus B_{r-1}$  of  $G$  of positions which can be reached with  $r$  moves but not with  $r-1$  moves. It is in some sense a “wave front” in the graph game. It defines the **BFS layer profile**  $\{|S_r(x)|\}_{r=0}^{\text{God}(G,v)}$ , the sizes of these fronts. Natural questions pop up: what is the structure of this sequence of integers? What is the value of the largest non-zero entry? Note that in the case of Cayley graphs  $\text{God}(G, v)$  is independent of  $v$  but for general game graphs, this is not necessarily the case. The BFS layer profile often appears unimodal (concave down): it increases then decreases.

**6.14. Examples.**

**1** For the sliding puzzle with  $\Gamma = K_n$ , the reachable configuration space from the identity is all of  $S_n$  and  $\text{God}(G) = n - 1$ . For  $K_5$ , the layer sizes are 1, 4, 12, 30, 44, 26, 3, totaling to  $120 = 5!$ . In this case, the BFS layer profile is independent of  $v$ .

The last layers of complete graph puzzles are harder to describe cleanly but show their own structure. For  $K_5$ , the last layer has exactly 3 configurations, all of cycle type  $(2, 2, 1)$ , and all with the hole at vertex 0. For  $K_6$ , the last layer has 35 configurations, split between cycle types  $(2, 2, 2)$  (with 15 representatives) and  $(3, 2, 1)$  (with 20 representatives), with hole positions distributed across all vertices. For  $K_5$ , the last layer is a single incomplete conjugacy class: it contains 3 of the 15 permutations in  $S_5$  with cycle type  $(2, 2, 1)$ , and all 3 have the hole at vertex 0.

**2)** For the sliding puzzle on the path graph  $\Gamma = P_n$ , each connected component of the game graph  $G$  is again a path  $P_n$ , so that  $\text{God}(G) = n - 1$ . In this case however, the maximal  $\text{God}(G, v)$  depends on  $v$ . If the initial condition is near the middle, then The BFS profile from the case when the hole is at the boundary is  $1, 1, 1, \dots, 1$ .

**3** For  $\Gamma = C_n$ , the game graph is  $G = C_{n(n-1)}$ . The last layer  $L_{\text{God}}$  contains exactly one configuration, the antipodal point.

**4** We can build **game graphs** coming from transposition games on trees, for which the BFS layer profile is not concave. Just take a tree with small vertex degree in distance  $\leq 3$  then have very large vertex degree in distance  $\leq 6$  and finally again small vertex degree until maximal radius 9.

## 7. CHESS GAMES

**7.1. Chess** is without doubt one of the most iconic multi-player games. It motivated the first steps in mathematical game theory [81, 72]. Many chess variants have been developed [61]. In the graph theoretical setup used here, we assumed that a game event is modeled by a finite simple maximal graph that starts at  $v$ . This obviously is not the standard assumption in chess, as one does not want to have the same position appear 3 times. And it does not matter for the God number, nor for the **Zermelo function**  $Zermelo : \{Games\} \rightarrow \{Vwins, Wwins, draw\}$ .

**7.2.** Let us discuss the repetition case issue. Assume  $G(n)$  is a game variant, where the game rule assumes that a position can not occur more than  $(n - 1)$  times and where otherwise, all rules are the same. Game events in a sub-graph  $H$  of  $G(n)$  are now the maximal finite simple paths in  $H$  which can not have more than  $n - 1$  self intersections. We formulated the result for chess but obviously it takes place for all games covered by the axioms. The following lemma is therefore a justification why the assumption that game events are finite simple maximal paths starting at  $v$  is a reasonable one and would not change essential things like "who wins" if one would allow a number of repetitions (like in chess).

**Lemma 3** (Repetition).  $God(G(n)) = God(G(1))$  and  $Zermelo(G(n)) = Zermelo(G(1))$ .

*Proof.* Assume  $G(n)$  is win for the initial player  $V=White$ . This means that  $V$  has a strategy  $H$  such that no game event for  $W$  with  $n - 1$  possible self intersections ends up in the winning set  $A$ . But then also no game events  $W$  that has no self intersections ends up in the winning set  $A$  so that also  $G(1)$  is win.

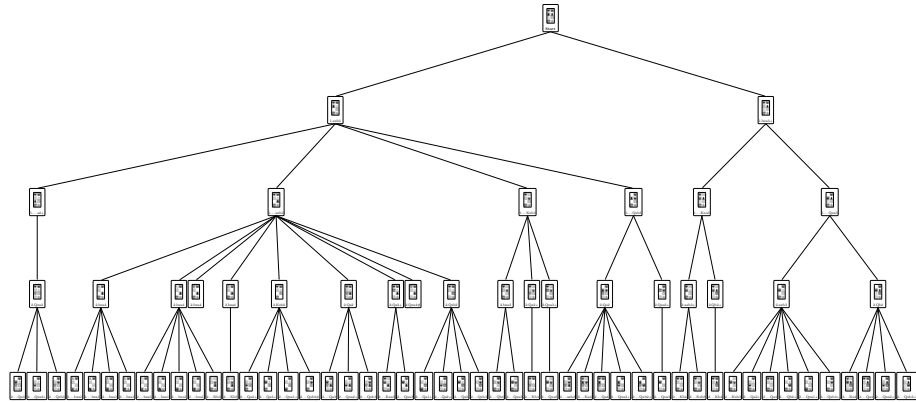
The same argument holds if  $G(n)$  is a win for  $W$ .

If  $G(n)$  is a draw, then there is a game strategy  $H$  to which both players have contributed such that there is no game event with possibly up to  $(n - 1)$  self intersections that ends up in  $A$ . So, also no game event with no self-intersections ends up in  $A$  which means that the game strategy  $H$  is also a game strategy for draw in  $G(1)$ .

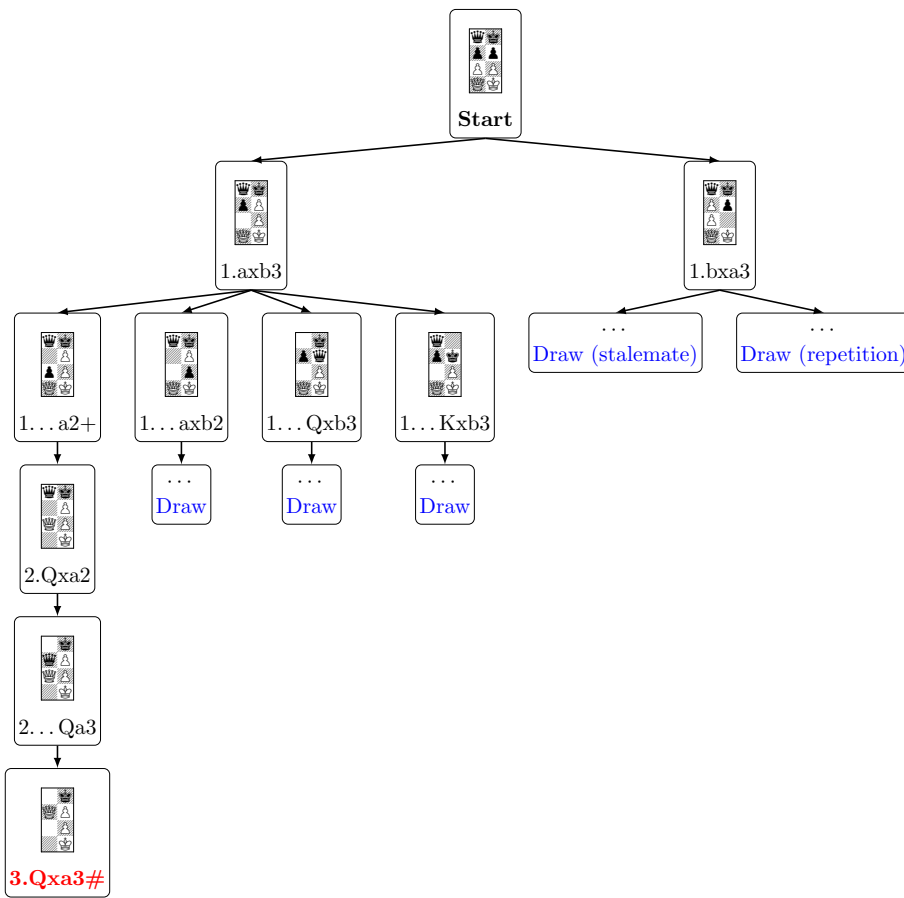
The opposite direction follows from Zermelo's theorem and logic: assume that  $G(n)$  is not a win for white, then this means that  $G(n)$  is a draw or win for black for  $G(n)$ . But in both cases this leads to a draw or win for black for  $G(1)$  and so not to a win for white for  $G(1)$ . The other cases are similar.  $\square$

**7.3.** We see that the theoretical simplification to assume finite simple paths as game events is also in very concrete situations like chess irrelevant. It does not change the basic question whether the game is a win or loss. Not imposing any limit on the number of times that a position can be revisited, would change things however. Such objections had been voiced to Zermelo's original approach. Having not only a finite set of game positions but also a finite set of game events is essential.

**7.4.** If we play on smaller boards, we can hope to explore the entire game graph. Here is the state graph of level 4 of the  $4 \times 2$  chess board.



7.5. One path leads down to a white forced win, but the other paths all lead to draws or stalemates assuming all players play rationally.



7.6. The  $4 \times 2$  chess board has pieces  $\{K, Q, P\}$  for both white and black players. It defines a directed digraph  $G = (V \cup W, E)$ . The graph starts from the starting position  $v \in V$ . The graph is generated using a breadth-first search (BFS) starting from  $v$ . For each state dequeued, we compute the legal moves using pseudo-legal move generation followed by check validation to ensure legality. States are added to the vertex set  $V \cup W$  and the queue only upon first discovery so that each unique position is processed exactly once. The BFS traversal guarantees that  $G$  contains all reachable states and represents the complete game tree. The height of a node is the distance to  $v$ .

7.7. The set of end positions  $S$  are the nodes for which the out-degree is zero. These are positions, where no legal continuation is possible like with **chess mate** or where the game is pointless to

continue, like of both parties only have kings. Within  $S$  is the set,  $A \subset S$  is the set of win positions. Game events are modeled as maximal simple paths starting at  $v$ . A path ending in  $A \cap W$  constitutes a victory for  $V=White$ , whereas a path ending in  $A \cap V$  constitutes a victory for  $W=Black$ . Terminal nodes that fall into the complement  $S \setminus A$ , which are draw positions. This can be **stalemates**, where the active player is not in check but possesses no legal moves. It is custom to include in  $S$  also situations, where a continuation is pointless as no win is possible. It is also assumed that  $A$  contains situations, where a continuation is hopeless. Nobody would battle out a game where a rook and a king battle a lonely king, even so the game path would allow to reach a stalemate still in principle.

**7.8.** Let us look at the following initial configurations: (I) king-side pawn only, (II) queen-side pawn only, and (III) where both pawns present. The corresponding graphs exhibit the following properties:

- Configuration I:  $|V \cup W| = 517$ ,  $|E| = 1034$ ,  $|A \cap W| = 10$ ,  $|A \cap V| = 11$ ,  $|S \setminus A| = 52$ ,  $\max d(x) = 16$
- Configuration II:  $|V \cup W| = 519$ ,  $|E| = 1037$ ,  $|A \cap W| = 10$ ,  $|A \cap V| = 11$ ,  $|S \setminus A| = 52$ ,  $\max d(x) = 16$
- Configuration III:  $|V \cup W| = 1966$ ,  $|E| = 4534$ ,  $|A \cap W| = 54$ ,  $|A \cap V| = 44$ ,  $|S \setminus A| = 113$ ,  $\max d(x) = 20$

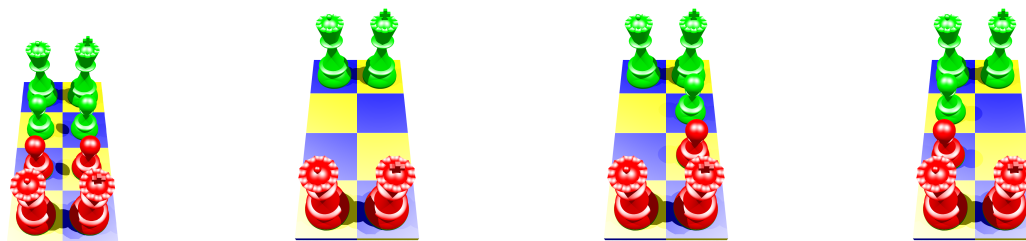


FIGURE 20. The middle two cases cases are “draw in one”. The strategy for  $V$  is to eliminate all but the capturing move. In the right,  $W$  wins.

**7.9.** The exponential growth in graph complexity from configurations I/II to III demonstrates the combinatorial explosion inherent in chess variants as piece density increases. The maximum shortest-path distance  $\max d(x)$  reflects the diameter of the reachable graph and indicates the minimum number of plies required to reach the most strategically distant positions. While the both-pawn configuration exhibits greater graph depth due to increased tactical complexity, determining the formal God number of these configurations - defined as the minimal graph path length evaluating over all sub-graph strategies will require further minimax optimization across the sub-graphs  $H \subset G$  to minimize the maximal potential loss. In theoretical analysis of chess, one usually does not consider **promotion** as this can render the game graph even larger.

**7.10.** Lets look at two small  $4 \times 4$  and  $2 \times 4$  chess games. In the  $4 \times 4$  situation there is mate in 3. In the  $2 \times 4$  game, there is a draw. Can we estimating the longest possible game, meaning the longest possible game event in the graph? In small situations as such it might be possible. A trivial upper bound for the game events is  $|V|^{|V|}$ , where  $V$  are the set of all possible positions.

**7.11.** Reducing the number of positions produces to **chess problem**, A typical problem gives the god number and asks to verify that this is the case.

**7.12.** Here is the  $2 \times 4$  game without paws after 2 moves. This was generated in Mathematica:

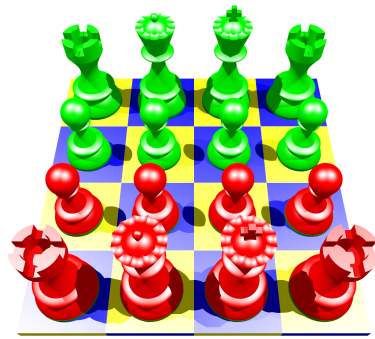


FIGURE 21. A  $4 \times 4$  board.

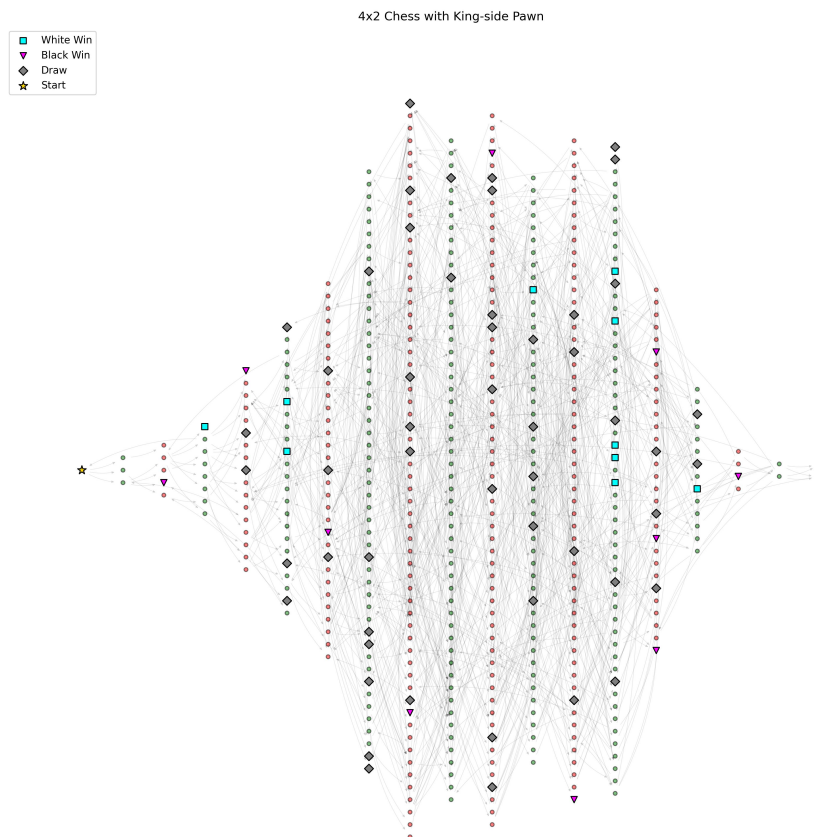


FIGURE 22. The game graph for the king side pawn.

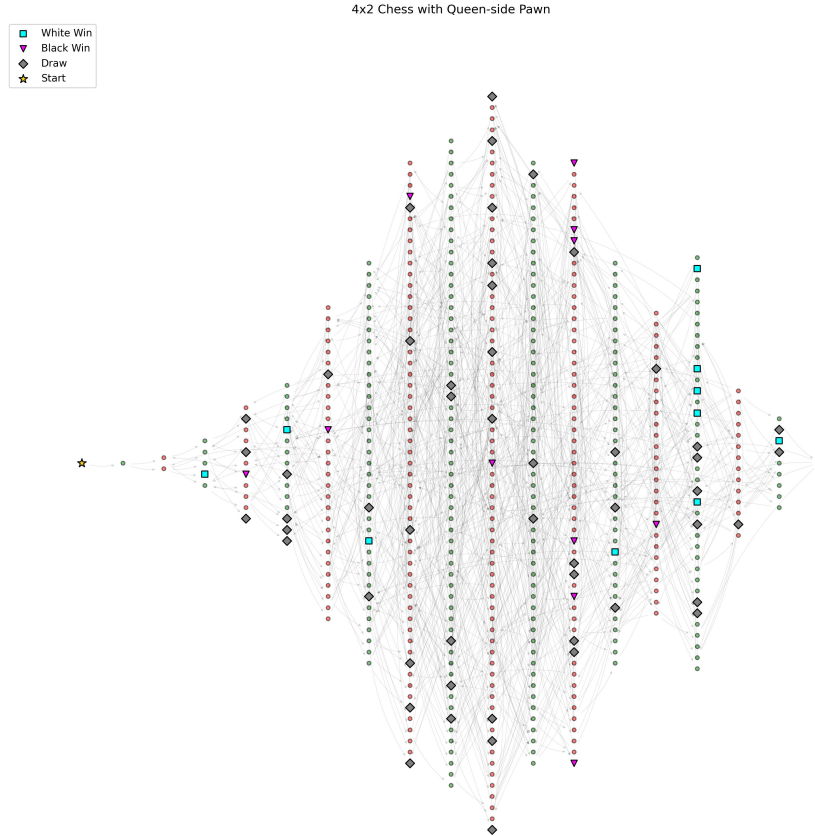


FIGURE 23. The game graph for the queen side pawn.

4x2 Chess

- White Win
- Black Win
- Draw
- Start

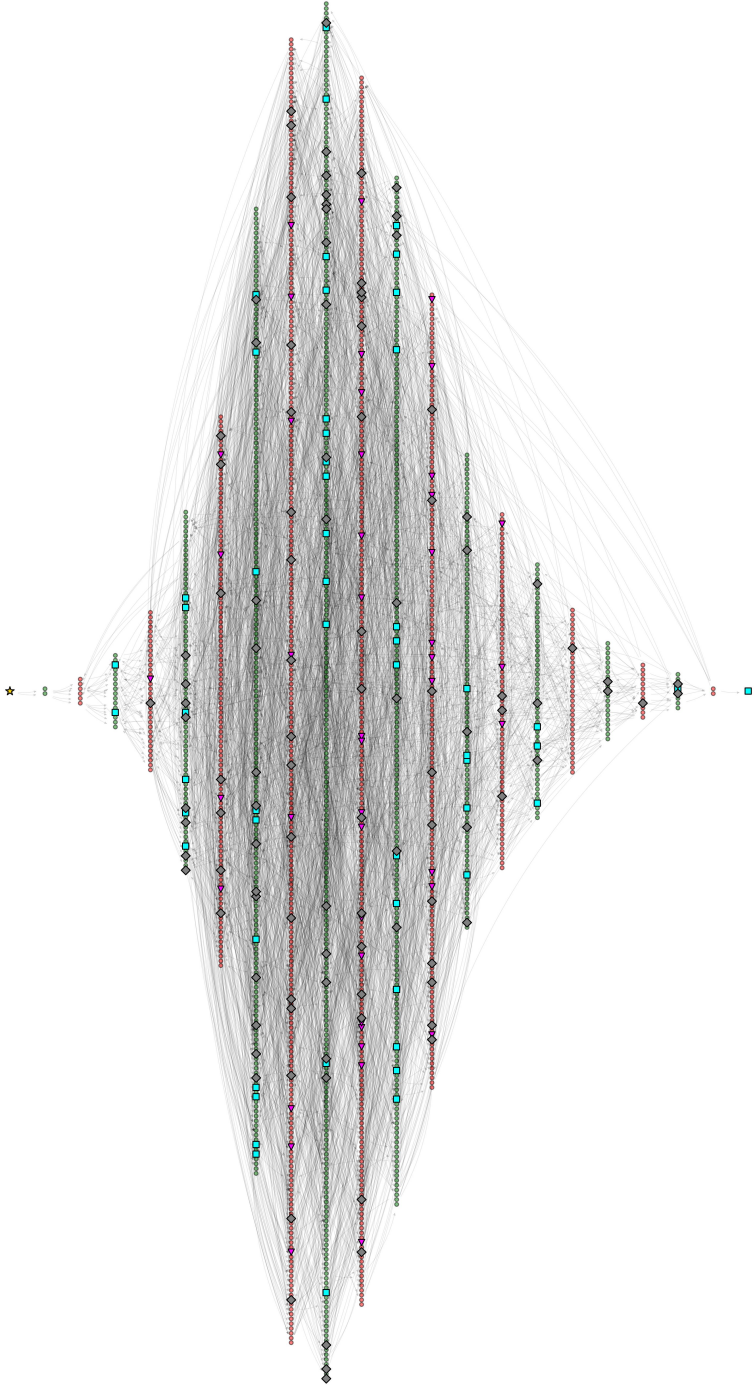


FIGURE 24. The game graph for both pawns

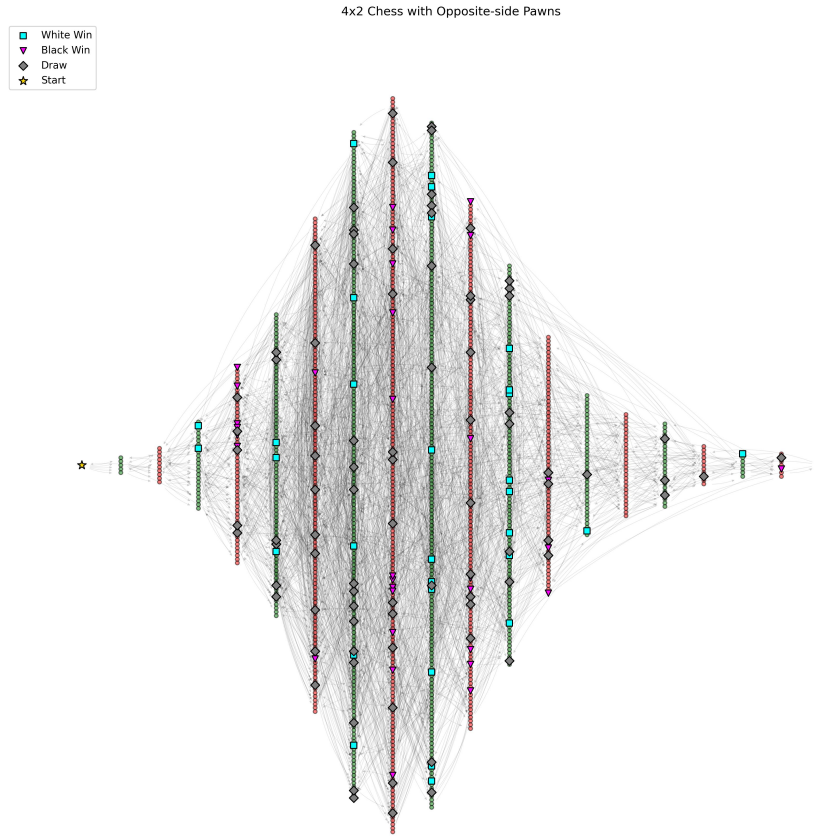


FIGURE 25. The game graph for opposite pawns

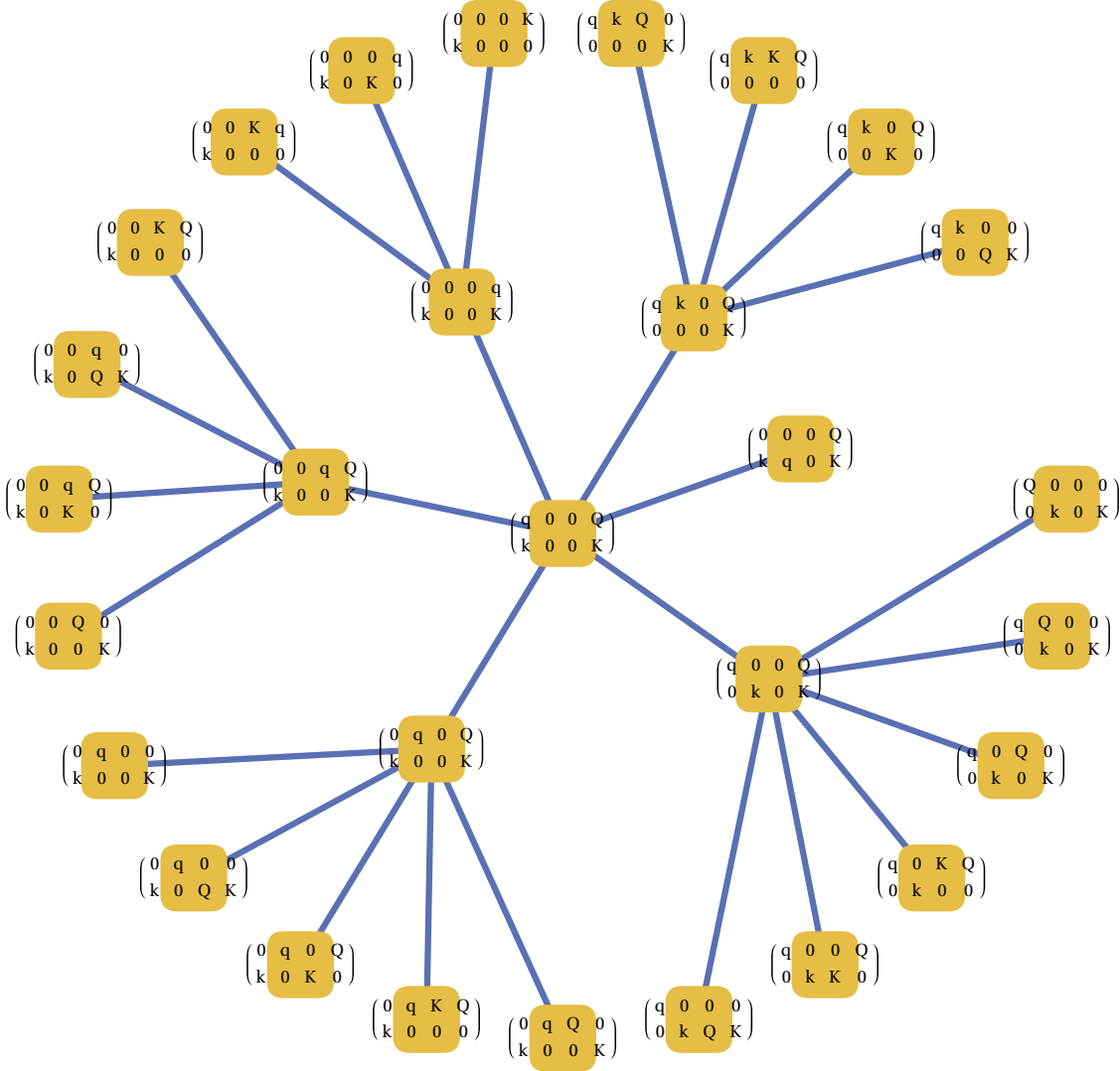


FIGURE 26. The ball  $B_1(v)$  of chess on a  $2 \times 4$  board without pawns. This game is "mate in 1". The queen can move in front of the king and force a chess mate. Lets call this position  $w$ . The winning strategy  $H$  of  $V$  is the subgraph  $H$  generated by the edge  $v \rightarrow w$ . All other edges are cut away. Player  $W$  would like to maximize the length of a game play in  $H$  but the edge degree of  $w$  is zero because it is a mate.

8. CARD GAMES

8.1. Consider a deck with two cards: ace and king. Each player gets a card. The one who has the ace wins. The game here is represented by a graph with two disconnected single points  $(V, E) = (\{A\}, \{K\}, \emptyset)$ . The **god number** is zero. The initial condition already is either win or lose.

8.2. Take a deck with 4 cards  $J, Q, K, A$ . Player V takes the first card, player B the second etc. The first one who has the ace wins. The  $V$ -winning situations are the permutations, where  $V$  is in an odd position. The  $W$ -winning situations are the permutations, where  $V$  is in an even position. The god number is 2.

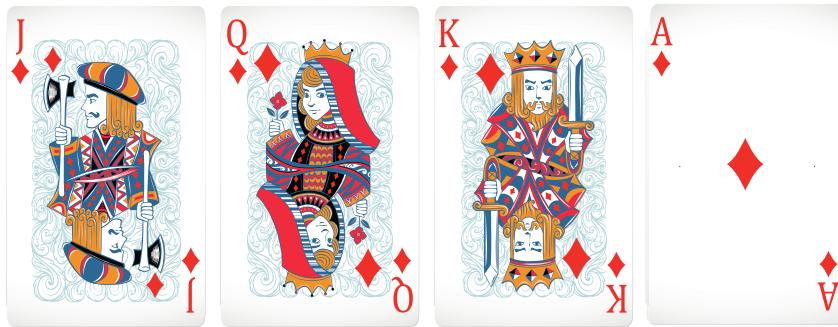


FIGURE 27. A 4 card game.

8.3. Is there a simple **Black Jack** 21 type game for which one has a chance to write down the full game graph? We experimented with an 11 version, where the highest card is 6 and only two suits. So there are 5 cards  $A_2, A_3, A_4, A_5, A_6$  and  $B_2, B_3, B_4, B_5, B_6$ . What happens with any game with random shuffle is that we need to look at all the possible permutation cases in which the card deck can be dealt. That is already for such small games heavy, like 10!

8.4. A simple 2-player card game is obtained by placing 3 pair of cards on the table. A player can flip two adjacent cards if the left is larger than the right. The cards are organized cyclic. The first player who can go from the initial condition 332211 to 112233 wins. There are  $6!/2^3 = 90$  different positions. Each player is allowed to switch identical cards during a game so that we break the parity. The graph diameter is 12, because 12 steps are needed to order the deck. We probably need to limit the number of identical switches. Variant: If we allow to switch any two cards (not only adjacent), the graph diameter is 4. We still have to see whether there is a winning strategy here, especially due to the passing. It could be that the parity decides.

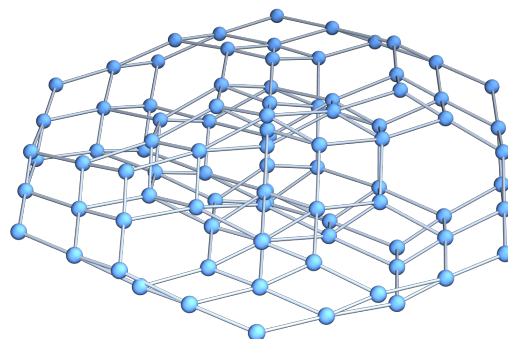


FIGURE 28. The graph of the card game with 90 positions.

**8.5.** To illustrate a 2-player card game, we use a variant of the transposition solitaire. Choose 2 queens, 2 kings and 2 aces from a standard 52 card deck. The initial position  $v$  is a random shuffle of this. A player can choose to transpose any two adjacent cards, noting that we utilize a cyclic order. The game ends if the state  $QQKKAA$  is reached. For example, with the starting arrangement  $QAKAQK$ , The legal moves are  $\{(QA), (AK), (KA), (AQ), (QK), (KQ)\}$ .

**8.6.** The graph  $G$  of the game has  $\frac{6!}{2!2!2!} = 90$  nodes and edges  $E$  as the set of pairs  $(x, y)$  for which there is a  $\tau$  with  $\tau(x) = y$  or  $\tau(y) = x$ . Let us look at the neighborhood graphs with radius  $d$  and center  $QQKKAA$ .

Distance $d$	Number of nodes
0	1
1	3
2	9
3	19
4	29
5	20
6	9
Total	90

**8.7.** Because every node has been reached after 6 steps, the god number of the game is 6. Only very close to the solution, there is a win. There are three possible initial conditions for which the first player gets to a win. The winning initial shuffles for player 1 to win are  $AQKKAQ$ ,  $QKQKAA$ ,  $QQKAKA$  In all other positions, the second player can always stir away from the win.

Outcome under Optimal Play	Number of Starting Shuffles
Player 1 wins	3
Player 2 wins	0
Draw	86
Total	90

**8.8.** The condition of being unable to repeat the transposition used on the previous term means the game is not solely determined by the arrangement of the cards, each player much also track which vertices were swapped. There are six possible remembered edges,

$$(1\ 2), (2\ 3), (3\ 4), (4\ 5), (5\ 6), (6\ 1)$$

together with one initial state in which no move has yet been played. Thus every arrangement contains one of 7 memory indicators. Therefore the full state space has size

$$90 \cdot 7 = 630 .$$

Each state contains not just the arrangement of the card, but also information on which player’s move it is, and which transposition was made on the previous turn. Computing every state gives the following results,

Type of expanded State	Count
Winning for player to move	18
Losing for player to move	7
Draw	605
Total	630

**8.9.** The hexagon's dihedral group ( $D_6$ ) works on our initial 90 configurations through rotations and reflections the board. Under these transformations, our 90 positions fall into exactly 11 symmetry classes,

Orbit Size	Number of Orbits
6	7
12	4
Total Orbits	11

We see that many positions are geometrically equivalent.

## 9. RUBIK TYPE GAMES

**9.1.** For group games and in particular Rubik type games, we work in a **finitely presented group**  $\langle S|R \rangle = \langle g_1, \dots, g_m | r_1 \dots r_m \rangle$ , where  $S$  are the **generators** and  $R$  are the **relations**. A group can also be defined by giving a finite set of generators in a permutation group. This is the **permutation representation**. One usually assumes that the **generator set**  $S$  is invariant under involutions. This gives god number 20 for the Rubik cube.

**9.2.** Many geometric shapes have been made into Rubik type puzzles in particular various  $n \times m \times k$  generalizations, or tetrahedra, octahedra or dodecahedra. There are version where the coloring has changed like **bi-colored cubes** [51]. One of the famous examples is the  $2 \times 2 \times 2$  Rubik cube already described in [74]. (it is also known as the “pocket cube”) and where we only allow moves of 180 turns of two faces L and U. This group has 6 elements and is isomorphic to  $\langle a = (1, 2), b = (2, 3) | a^2 = b^2 = (ab)^3 \rangle = S_3$ . The god number is 3. For the full pocket graph, the Cayley graph can be constructed by GAP and the god number determined. See the code section.

**9.3.** The  $2 \times 2 \times 1$  Rubik is called the “Floppy”. Its group has 2304 elements. The god number is 13. If we work with the floppy, where one of the cubes is fixed, the number of positions drops to 192 and the God number is 6. There are  $k \times l \times n$  versions of Rubik cubes as seen in Figure 3.5.

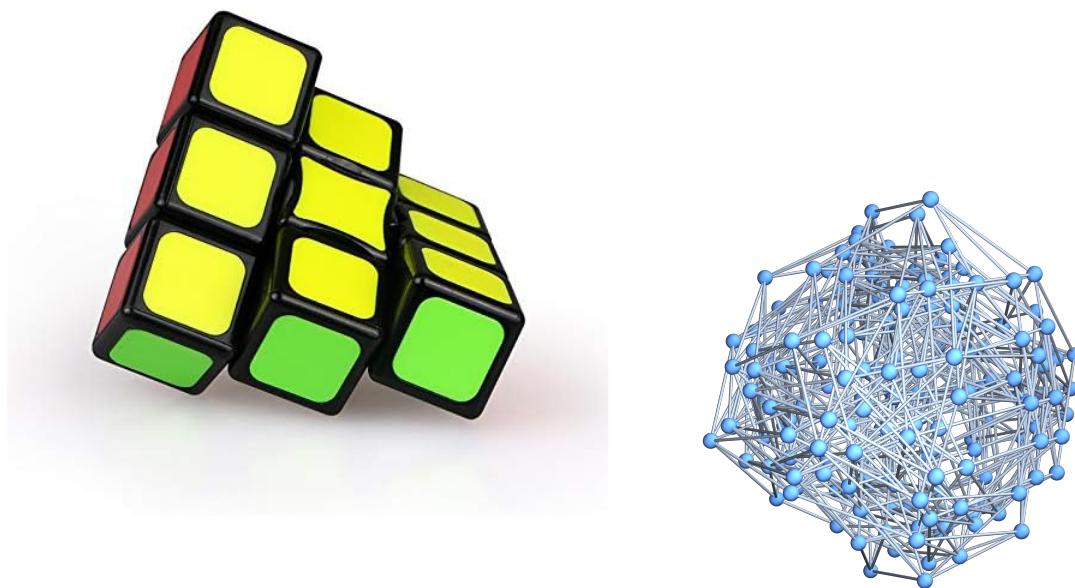


FIGURE 29. The **floppy** was designed by Katsuhiko Okamoto.

**9.4.** For the  $2 \times 2 \times 2$  cube case, the god number is 11 respectively 14 is known. There are  $3'674'160$  positions for the  $2 \times 2 \times 2$  (pocket) cube in the half turn metric the god number is 11 (analog to 20 in the standard Rubik  $3 \times 3 \times 3$  cube). If one of the cubes is fixed, the number of positions reduces to  $8!3^7$  which is  $1/24$  of  $3'674'160$ . In the quarter turn metric, the god number of the pocket cube is 14 (this is the analog to the god number 26 in the  $3x3x3$  case). The GAP code in the code section was able to get this number within a few hours of run time. The subgroup of the full  $3 \times 3 \times 3$  Rubik generated by 180 degree points is a group with 663552 elements and has god number 15.

**9.5.** Simpler and older than Rubik are interlocked rings, called the **Hungarian rings**. There are two cyclic permutations generating a subgroup of the permutation group such that the two cycles intersect in 4 points. There are  $38!$  possible positions. Gap was unable to compute the diameter in this case. Smaller Hungarian rings can be done.

```
LoadPackage("grape");
a:=(1,38,37,36,35,6,34,33,32,31,30,29,28,27,26,25,24,23,22,21);
b:=(1,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2);
G:=Group(a,b); Size(G);
S:=GeneratorsOfGroup(G); C:=CayleyGraph(G,S); Diameter(C);
```

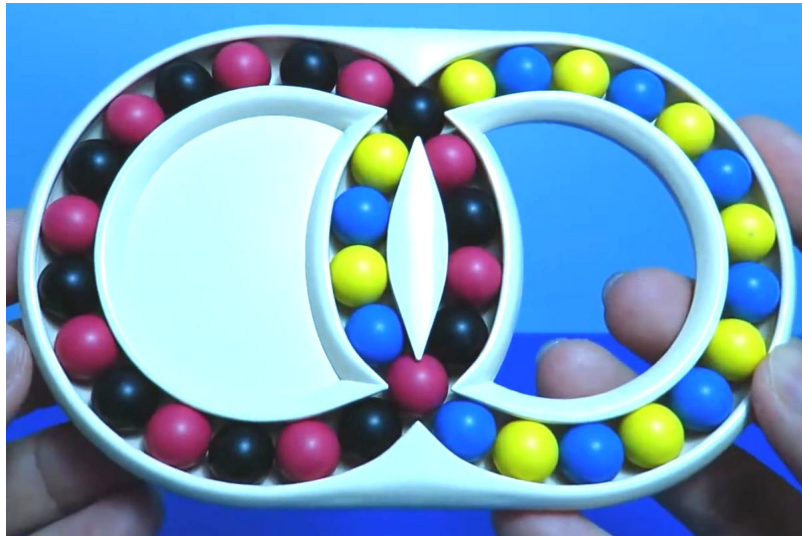


FIGURE 30. Hungarian rings

**9.6.** For the **Rubik's clock**, the god number is known to be 12. There are  $2^{14}$  positions. The Rubik's clock has lots of generators: there are  $2^4$  flag positions and in each case 4 generators. So that there are  $2^6 = 64$  generators.



FIGURE 31. Rubik’s clock has many generators.

### 10. TIC TAC TOE GAMES

**10.1.** The standard  $3^2$  Tic-Tac-Toe is famously known to be a draw [28]. The  $n^d$ -game looks for a line of length  $n$  in a  $d$ -dimensional hypercube of side  $n$ . The two-dimensional  $m, n, k$ -**game** ( $k$  in a row on an  $m \times n$  board) formalise positional tic-tac-toe [10]. Hales and Jewett [30] used strategy-stealing to show no second player wins an  $n^d$ -game, and that  $n^d$  is a first-player win, once  $d$  is large relative to  $n$  (like  $d > 3^n$ ). A particular example mentioned in [30] is **Gomoku**, which is a **5 in a row game**. It is usually played on a  $15 \times 15$  or earlier as  $19 \times 19$  board. <sup>9</sup> Patashnik [?]olved the  $4^3$  "Qubic". The first-player wins Allis [3, 4] showed free-style Gomoku (15, 15, 5) is a first-player win. Already the  $5 \times 5 \times 5$  Tic-Tac-Toe with  $3^{125}$  positions is unsolved [7]. [35] computationally established (7, 7, 5) and (8, 8, 5) as draws.

**10.2.** For  $n^d$  games, the following is known:

Size	God’s number	First-player win?	Reference
$1^d$	1	Yes	trivial
$n^1, n \geq 2$	—	No (draw)	trivial
$2^d, d \geq 2$	3	Yes	folklore (strategy-stealing)
$3^2 = (3, 3, 3)_{2D}$	— (draw at move 9)	No (draw)	[5]
$3^3$ (3D, $3 \times 3 \times 3$ )	not precisely established	Yes	[8, 3]
$4^3$ (Qubic)	not precisely established	Yes	[7, 8]
$n^d, d \gg n$	—	Yes	[6]
$n^d, n$ odd, $d \leq f(n)$	—	No (draw, pairing)	[6]

**10.3.** For two dimensional  $m, n, k$ -games, the God’s number is given in brackets:

<sup>9</sup>Gomoku means five pieces in Japanese. It is also called **Wuziqi** in China or **Omok** in Korea of **Gobang** in Britain



FIGURE 32. Connect 4 game.

Board	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$3 \times 3$	Win (2)	Draw	—	—	—
$4 \times 4$	Win (2)	Win (3)	Draw	—	—
$5 \times 5$	Win (2)	Win (3)	Draw	Draw	—
$6 \times 5$	Win (2)	Win (3)	Win	Draw	—
$7 \times 7$	Win (2)	Win (3)	Win	Draw	Draw
$9 \times 6$	Win (2)	Win (3)	Win	Draw	Draw
$15 \times 15$	Win (2)	Win (3)	Win	Win (Gomoku)	Draw

**10.4.** From [7]:  $k = 1, 2$  are trivial wins except for  $(1, 1, 2)$  and  $(2, 1, 2)$ ;  $(m, n, 3)$  is a win if and only if  $(m \geq 4, n \geq 3)$  or  $(m \geq 3, n \geq 4)$ , else draw. First general result:  $(m, n, k)$  is a draw whenever  $m = n = k \geq 3$ .  $(m, n, 4)$ : draw for  $m, n \leq 5$ ,  $(6, 5, 4)$  win,  $(m, 4, 4)$  win if and only if  $m \geq 9$ ;  $(m, n, 5)$ : draw for  $m, n \leq 8$ ;  $(15, 15, 5)$  win,  $(9, 6, 6)$  and  $(7, 7, 6)$  draw via pairings.

**10.5.** Tic-Tac-Toe Games are also played **with gravity**: impose a vertical axis and require pieces to stack from the bottom of each column. It is the  $n^d$  analogue of Connect Four. Strategy-stealing still rules out second-player wins on  $n^d$ , but values change for some 2D  $m, n, k$  sizes. God numbers also can be given in draw situations. In the  $3^2$  case, a draw reached at move 9 as the board is filled.

Size	G	H (gravity)	God(G)	God(H)	Ref
$1^d$	Win	Win (identical)	1	1	trivial
$n^1, n \geq 2$	Draw	Draw (moves forced)	—	—	trivial
$2^d, d \geq 2$	Win	Win (corner: $2^{d-1}$ threats)	3	3	folklore
$3^2$	Draw	Connect-3 on $3 \times 3$ ; not in Tromp's tables	9	?	—
$3^3$	Win	Connect-3 on $3 \times 3 \times 3$ ; status open	?	?	—
$4^3$ (Qubic)	Win [8]	Open: Connect-Four 3D on $4^3$ unsolved [10]	?	?	[10]
(5, 5, 4)	Draw	Draw	—	—	[9]
(6, 5, 4)	Win	Draw	?	—	[9]
(6, 6, 4)	Win	Second-player win	?	?	[9]
(7, 6, 4) Connect 4	Win	Win, forced by move 41	?	41	[1, 12]

**10.6.** Gravity preserves the god's number for  $1^d$ ,  $n^1$ , and  $2^d$ , but can decrease the number of playable, winnable lines for the first player. Qubic without gravity is a first-player win [60]. With gravity ( $4 \times 4 \times 4$  **Connect-Four 3D** was solved independently by Allen and Allis in Oct 1988. For  $7 \times 6$ ), it remains open. Gravity can even flip (6, 6, 4) from a first-player win to a second-player win

**10.7.** For  $q = 2$ , we have looked at the  $2 \times 2$  case, which is win for the starter with god number 2. The case  $3 \times 3$  which is always a tie with  $\text{ceiling}(n^2/2)$  god number. For  $3 \times 3 \times 3$ , there is always win.

11. HANOI PUZZLES

**11.1.** The **tower of Hanoi**  $T_3(n)$  was posed by Edouard Lucas in 1883 with the legend of 64 golden disks [48]. In this three-peg  $p = 3$  situation, the god number is  $2^n - 1$ . It is a standard example in many programming textbooks like [66]. Dudeney [21] introduced the **Reve’s puzzle**  $T_4(n)$ , which is the four-peg version. Frame and Stewart [24, 76] independently proposed a divide-and-conquer recurrence for  $p \geq 4$ , later shown equivalent to seven other algorithms [33]. [17] (building on Szegedy 1999) established the correct order of magnitude. Bousch [12] proved that Frame-Stewart is optimal for  $p = 4$ . We see below the graphs for  $p = 4$  and  $n = 2, 3, 4, 5, 6$ . Optimality for  $p \geq 5$  is open. See the book [33].

Variant	$T_p(n)$ / god’s number	Status	Reference
Lucas 1883 $p = 3$	$2^n - 1$	Proven	[48, 33]
Reve’s puzzle, $p = 4$	Frame–Stewart recurrence	Proven	[12]
$n = 1$	1, 3, 5, 9, 13, 17, 25, 33, 41, 49	Proven	[56]
Multi-peg, $p \geq 5$	Frame–Stewart recurrence	Conjectured	[24, 76, 17]
F-S formula	$F(p, n) = \min_{1 \leq i < n} \{2F(p, i) + F(p-1, n-i)\}$	—	[33]
asymptotic order	$2^{(1 \pm o(1)) (n(p-2)!)^{1/(p-2)}}$	Proven	[17]
Cyclic (directional) Hanoi, $p = 3$	exponential, closed recurrence	Proven	[6]
Arbitrary-to-arbitrary, $p = 3$	diameter of $H_3^n$ still $2^n - 1$	Proven	[9]
Arbitrary-to-arbitrary, $p \geq 4$	diameter unknown in closed form	Open	[9]

**11.2.** The game graph is the **Hanoi graph**  $H_p^n$  in which the vertices are the  $p^n$  strings in  $\{1, \dots, p\}^n$  encoding the disk  $i$  on peg  $s_i$ . The order on each peg is forced to be decreasing in size) and edges connect configurations differing by one legal move. The diameter of  $H_3^n$  is  $2^n - 1$ , attained between any two perfect states [33].  $|V(H_3^n)| = 3^n$  and  $|E(H_3^n)| = \frac{3(3^n - 1)}{2}$ . [71].

**11.3.**  $H_3^n$  is three copies of  $H_3^{n-1}$  joined at their extremal vertices by bridging edges [33]. It is planar, a penny graph, and its drawing converges to the Sierpinski triangle [32]. The vertex set coincides with the odd entries of the first  $2^n$  rows of **Pascal’s triangle**. Average distance, chromatic, and metric properties are given in [33]. The graph  $H_p^n$  has  $p^n$  vertices; its diameter is unknown for  $p \geq 4$ . For  $p > 4$  with  $n > 0$ , or  $p = 4$  with  $n > 2$ ,  $H_p^n$  is non-planar [34]



FIGURE 33. The tower of Hanoi with n pieces has god number  $2^n - 1$

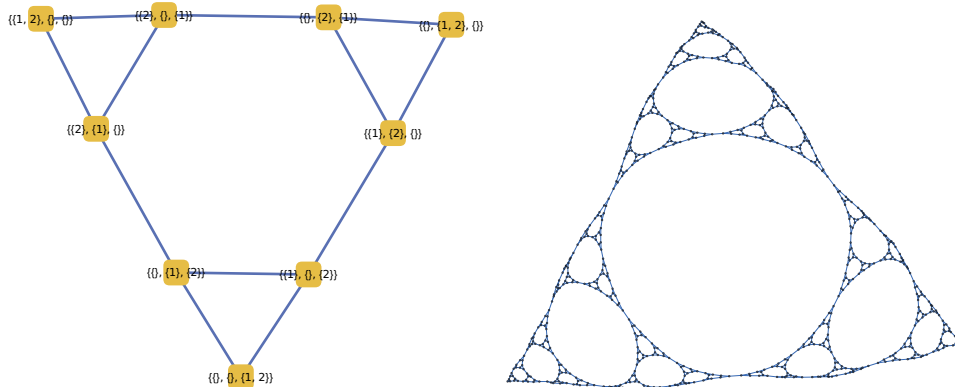


FIGURE 34. The Hanoi graph in the case  $n = 2$  has 9 vertices, 12 edges and diameter 3. We see also the graph for  $n = 6$ , where we have  $|V| = 729$  and  $|E| = 1092$ . The positions where all disks are on one peg are on the boundary.

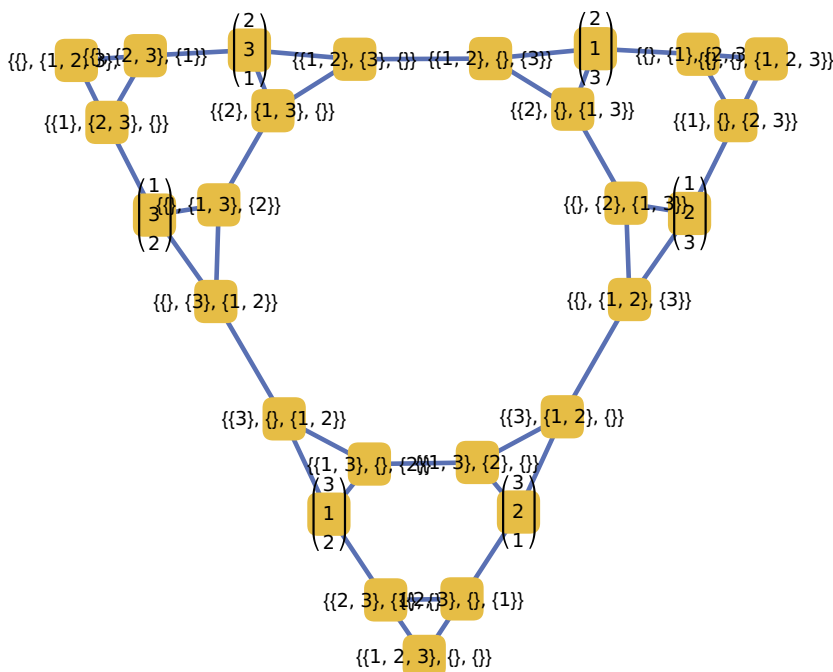


FIGURE 35. The Hanoi graph for  $n=3$ . Going along the boundary is the optimal path. It recursively first uses the solution for  $n=2$ , then moves the entire pile and then again uses the solution for  $n=2$ .

**11.4.** The Hanoi problem with  $p = 4$  pegs is not explored for large  $n$ . The 4 peg problem is a more interesting transport problem as one can now work more effectively using two "storage pegs". The game graph obviously has a  $S_4$  symmetry, which leads to a tetrahedral symmetry when plotted in 3- dimensional space  $\mathbb{R}^3$ .

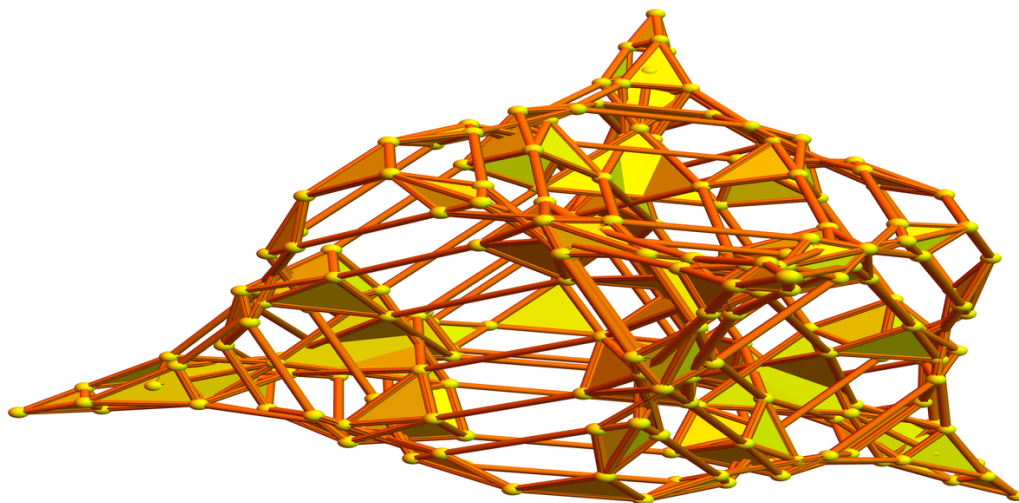


FIGURE 36. The graphs of the Reeves puzzle, ( $p = 4$ -peg Hanoi puzzle) have smaller diameter as we have more transport possibilities. We measure the graph diameters  $\text{God}(2) = 3, \text{God}(3) = 5, \text{God}(4) = 9, \text{God}(5) = 13, \text{God}(6) = 17$ . We see here the case  $n = 4$ .

## 12. OTHER GAMES

**12.1. Hex** [31] is a two-player game that has relations to theorems in topology. It can be played on any rectangle [50]. It was invented in 1942 by Piet Hein and rediscovered by John Nash in 1948. Hex is a paradigm because of the **Hex theorem** telling that it never leads to a draw. It has been used to prove the Brouwer fixed point theorem [25], who also pointed out that it is also related to the Jordan curve theorem. David Gale also covered the  $n$ -dimensional Hex theorem on a  $n$  dimensional cube which belongs to a  $n$ -person Hex game. Gale writes *From the game-theoretic point of view,  $n$ -person Hex is probably not of much interest. Aside from the mechanical problem of designing a practical board there is the usual difficulty of multi-person games that players may form coalitions, requiring more sophisticated analysis how such coalitions can be formed and play against each other.*

**12.2.** Hex type games can be played on any finite simple graph  $\Gamma$ . Each player selects out an edge. The first one who has a closed simple loop of the same color wins. A famous variant is a square grid on a torus. Player A needs to build a path from left to right not crossing any path of B. Player B needs to build a path from bottom to top to bottom. The most famous variant is "Hex" which has been popularized by Nash (but Nash would play on a square board, not a hexagonal lattice). It is an example where the first player has a winning strategy.

**12.3. Hex on a torus** is considered to be harder than the standard hex. Also here, the advantage of the first player can be balanced with the pie rule telling that after the first move the second



FIGURE 37. In hex, there is no draw.

player can decide whether to take over the position of the first. Also for Torus Hex there is no tie because of topology. Also here, The first player always wins because if the second had a strategy, it could be used by the first without the first stone. The topology is related to Brouwer fixed point theorem and Sperner lemma. God number is larger or equal to  $n$ . The  $3 \times 3$  Hex board is isomorphic to Tic-Tac-Toe. It is also related to the P-NP problem.

**12.4.** Related to Hex is the **Shannon switching game**. As Martin Garner reported in one of his Scientific American puzzles and diversions articles, the game has first been described by Claude Shannon, in the early 1950ies. Take an arbitrary graph  $\Gamma$  in which two subsets  $A, B \subset V(\Gamma)$  are selected. Two players take turn coloring the edges  $E(\Gamma)$  of  $\Gamma$ . The first one who can build a connection between  $A$  and  $B$  wins. When played on a rectangular grid, it is the game **Bridg-It** described by David Gale a few years after Shannon and this was also described in one of the Scientific American columns. The players are usually denoted as  $V$ ="Short" and  $W$ ="Cut". While "Hex" is played on vertices of a graph, the Shannon switching game is played on edges. Like Hex, there is always a winner in this game, no draw.

**12.5. Peg games** are **solitaire games**. The god number is explicitly known for some. The topology decides whether there is a win or not. Which topologies allow a reduction to a point? It can be NP-Hard. [77, 41]. In the case of a triangular peg game with side length 3, the game is a draw. There is simply no path to the solution where only one peg remains. The most common peg solitaire is probably the English board on 44 holes. The European board has 4 holes more. The general problem is called also  $Hi - Q$  and NP-complete.



FIGURE 38. A peg solitaire where we play on a triangular grid with 6 holes. The 9 holes to the right are considered off grid. In this case the game has no solution with this initial condition  $v$ , where  $A$  is the set of configurations with one peg.

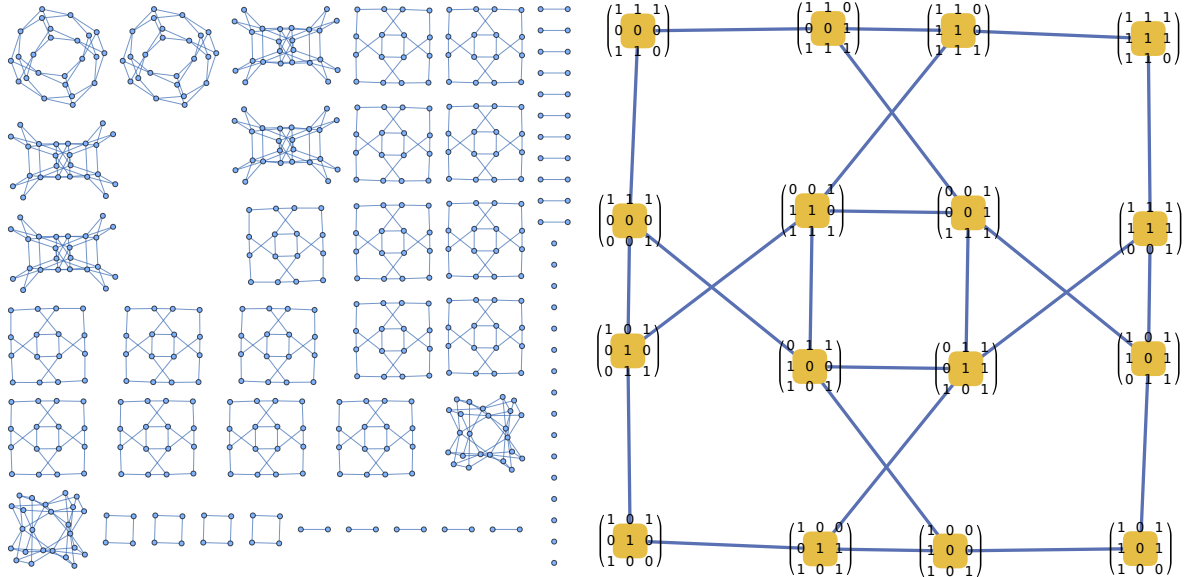


FIGURE 39. To the left we see the full game graph on a  $3 \times 3$  lattice Peg solitaire. It has  $2^9 = 512$  vertices. To the right we see the connected component of an initial condition with one peg missing (upper right vertex). The game can not be solved. The code producing this graph is below.

**12.6.** The 80ies were already in a frenzy about Rubik type games. There were Hungarian ring type games like the **Nintendo drum** in which 25 balls could be rotated around or then the **masterball** in which a sphere is cut into  $4 * 8 = 32$  pieces.<sup>10</sup> The user could rotate by 180 degrees along a longitudinal line or then by  $360/8$  degrees any of the 4 latitude layers. The permutation group has  $16!^2 = 437763136697395052544000000$  which is a subgroup of  $S_{32}$ .

**12.7. NIM** is 2-player game in which the goal is to take the last object. Each player can take an arbitrary number of objects from one pile. For two piles, there is a winning strategy for the first player if the pile sizes are different. Otherwise, the second player wins. If both piles have  $n$  pieces for example, then the god number is  $n$ .

**12.8.** Shannon writes in [72]: *“In some games, there is a simple evaluation function  $f(P)$  which can be applied to a position  $P$  and whose value determines to which category (won, lost, etc.) the position  $P$  belongs. In the game of NIM (Hardy and Wright, 1938), for example, this can be determined by writing the number of matches in each pile in binary notation. These numbers are arranged in a column (as though to add them). If the number of ones in each column is even, the position is lost for the player about to move, otherwise won.*

**12.9.** An other class of arithmetic games are **Zeckendorf games**. Like NIM or alpha-zero [8] it is based on arithmetic. The Zeckendorf game is based on the fact that every integer can be written as a sum of non-adjacent Fibonacci numbers. The now starts with a number and ends when the number is Zeckendorf decomposed. The game always ends as a win for one of the players. For example, for  $n=13$ , player 2 has a winning strategy. For  $n=32$ , player 1 has a winning strategy. [15] displays an example where a full game tree can be displayed.

**12.10.** For **checkers** we have a forced draw. A position with 24 kings could be reached in 211 moves. On a  $10 \times 10$  board, a proof game can last up to 389 moves. On an  $n \times n$  board, it is PSPACE-hard to determine whether a specified player has a winning strategy.

<sup>10</sup>Also called GeoMaster. Not to be confused by much later considered Hanayama Polekmon puzzle



FIGURE 40. The masterball puzzle is a Cayley graph in the group  $S_{16} \times S_{16}$ . The puzzle is still sold on ebay (picture).

**12.11.** There are 3,6 or 9 Men's **Morris** versions. It is known **Mühle**=mill in German. In **9 Men's morris**, the players place and move nine men to form "mills" which are three in a row, allowing to remove an opponents piece. The game ends when a player is reduced to two pieces. The Nine Men's morris goes back to the Roman empire. This game is solved in the sense that we know it is a "draw" case in the Zermelo trichotomy. The Twelfe Men's morris adds four diagonal lines and is known as **Morabaraba**.

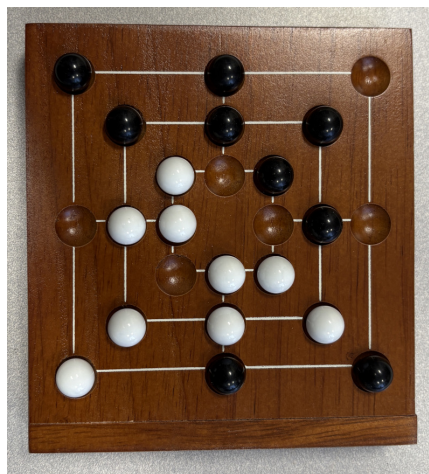


FIGURE 41. Nine Men's Morris is known to be a draw case in the Zermelo theorem.

**12.12.** **Mankala** is a 2-player game that is also known as **Serrata**. Its transport nature makes it resemble the more popular dice game **Backgammon** but it is completely deterministic. The

player who takes the last object loses. A game is encoded as a vector  $n = (n_1, \dots, n_k)$ . The parity of the binary digital sum  $n_1 \oplus n_2 \oplus n_k$  decides then, who wins. On smaller boards and less stones, one can write down the entire game graph.



FIGURE 42. Mankala is also known as Serrata. Unlike Backgammon, it is deterministic. Each player has pits and a bank to the right. The goal is to cash in as many stones as possible. To play, a player takes a pile from its own side and distributes the stones anti-clockwise, one by one. When ending in the bank, the player can play again. Play until no stones are any more in the pits. The game produces a complex game tree. As in chess, one can for small  $n$  and  $m$  ( $n$  pits,  $m$  stones initially in each pit), write down the game tree. For  $n = 2, m = 2$  for example, the first player wins.

**12.13. Tetris** leads to polyomino packing problems. These are solitaire games. Polynomios are also called **lattice animals**. Tetris problems are also a source of nice 1-player games. They are related to lattice animals (polyominos). The first interesting puzzle can be done with the **5 tetrominoes**. By area they could fit into a  $4 \times 5$  box but a checkerboard argument shows that we would need 11 colors of one color and 9 of an other because the T animal has 3 of one color and 1 of the other while all other animals have 2 of one color and 2 of the other. Famous are the 12 free pentominoes: there are 2339 ways to arrange them in a  $10 \times 6$  rectangle.

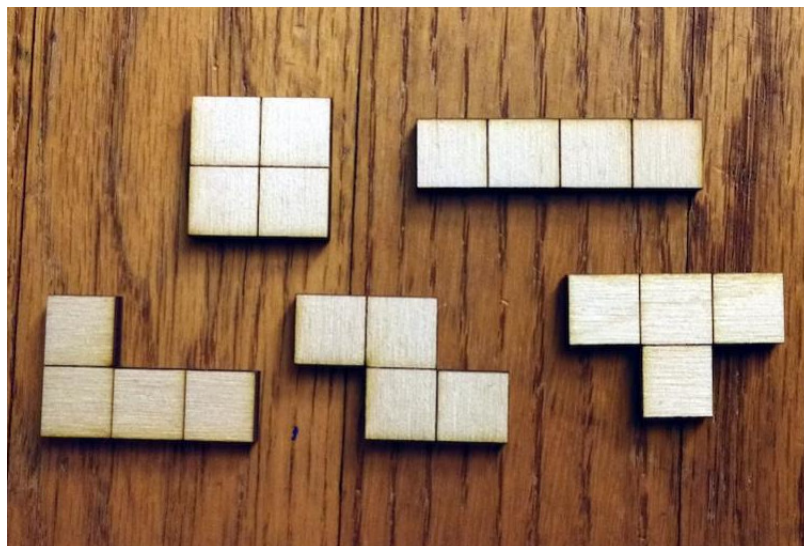


FIGURE 43. Tetrominoes

**12.14.** A more recent sliding game with larger hole size is the **cross**. It is a permutation sliding game on  $S_{13}$  with 4 holes so that a-priori only  $13!/4!$  positions have to be considered. Since there is ample of space the sorting can go quite efficient. There is a difficulty in that the balls are moved by gravity. One can naturally see this game however as group game as it can be seen as a finitely presented group with 4 generators. Each generator  $t, b, l, r$  (for top bottom, left right) tilts the game physically in space so that one of the ends of the cross is at the bottom. The generators satisfy  $tb = bt$  and  $lr = rl$ . A possible element in that presented group is  $tlttlbr$ . We have not yet looked at the exercise to find the relations in this finitely presented group.



FIGURE 44. Cross parking game. An example of a sliding game with more holes. (was bought at the MoMath Museum in New York).

**12.15.** New games are also developed in modern times. An example **alpha-zero** described in [8]. In one of the games there, one has a stack  $Z = \{1, \dots, n\}$ . The bank starts with a total 0. Player  $V$  picks a number from  $Z$  and either adds or multiplies it to the total. The number is removed from the stack. The second player now picks a number from the stack and adds or multiplies it to the total. The first player who reaches a total that is multiple of  $n$  wins. This can also be played with other stacks like a collection of cards and a given number  $n$ .

### 13. REMARKS

**13.1.** Conway constructed numbers as a subclass of games [18]. It is concept allowing to construct **surreal numbers**. The axiomatic setup of **Conway's combinatorial game theory** can describe any two-player game with perfect information and no chance. Donald Knuth wrote a novel [42] about it, in which two people found themselves on a deserted island with a stone telling some of the axioms of the concept. Knuth's title in English was *Surreal Numbers: How Two Ex-Students Turned On to Pure Mathematics and Found Total Happiness*. The German title of that book was **island of numbers**. Note however that even for simple games like Tic-Tac-Toe, one would hardly write the game as part of the "numbers of games" paradigm.

**13.2.** Many solitaire game can be made into a 2 player game where two-players take turn. The 6-card game we have seen is first of all a simple solitaire sorting game. The one who lands in the win position wins. This is in general not that difficult to analyze. Such games can be played on any Cayley graph. One could also give each player the option to lock one of the generators. For transposition games where  $G = S_n$  one can naturally take  $V = A_n$  the set of even permutations and  $W = S_n \setminus V$ , the set of odd permutations. After choosing  $v$  and  $A$  one has a game.

**13.3.** God number problems are related to **run length analysis** of algorithms, like sorting or path finding. A **breadth first search** builds up larger and larger neighborhoods. To get from a neighborhood of all points in distance  $r$ , build all unit balls from all vertices of the boundary to get a ball of radius  $r + 1$ . This is **Breadth-First Search**. Once all vertices are covered, we have the largest distance from  $x$ . This can be inefficient. It can be that a vertex  $x$  has  $S_r(x) = G$  but  $r$  is smaller than the spectral radius. A simple example is the wheel graph, where the center  $x$  gives  $B_1(x) = G$  but where the graph diameter is 2.

**13.4.** A general strategy to generate 2 player game is to take a graph, select a certain geometric property for partially 2-colored graphs like having a path graph of length 4 with one color. Then have two-players place colors on edges. The first one who realizes the pattern wins. Already in the solitaire game, finding patters is NP complete in general. Finding complete sub-graphs of a graph for example is NP complete.

**13.5.** The collection of all solitaire games could be made into a **category**. The objects are the graphs a homomorphism from  $(V, E, v, A)$  and  $(W, F, w, B)$  would be a map  $\phi : V \rightarrow W$  that maps edges to edges or vertices and which maps  $v$  to  $w$  and  $A$  to a subset of  $B$  and for which the inverse  $\phi^{-1}(y)$  is a connected sub-graph of  $(V, E)$  for every  $y \in W$ . Homomorphisms do not honor "win" however in general. Two games are **isomorphic** if there exist a graph isomorphism that maps  $v$  to  $w$  and  $A$  to  $B$ . Isomorphic games have the same god number. An example are the following two  $2x2x2$  Rubik implementations.

**13.6.** The disjoint union of two solitary games in which  $A$  in the first game is connected to the initial condition in the second. This is the disjoint union of the graphs. The god number is additive. If we add two Rubik's cubes for example, the god number to solve both of them is  $20 + 20 = 40$ . If we add two multiplayer games, the situation is more complicated. If player  $V$  decides to focus on one game,  $V$  will probably win that game if the other player focuses on the other game. An example of addition of games are competitions where many cubes have to be solved.

**13.7.** **Value systems** could allow to estimate the god number. This has been pointed out already in [72]. They can serve as the analog of **Lyapunov functions** in dynamical systems theory. If we can find a value system which has a maximum at the equilibrium set  $S$  and design a **gradient method** with respect to this function, then we have some hope that the gradient ascent will lead

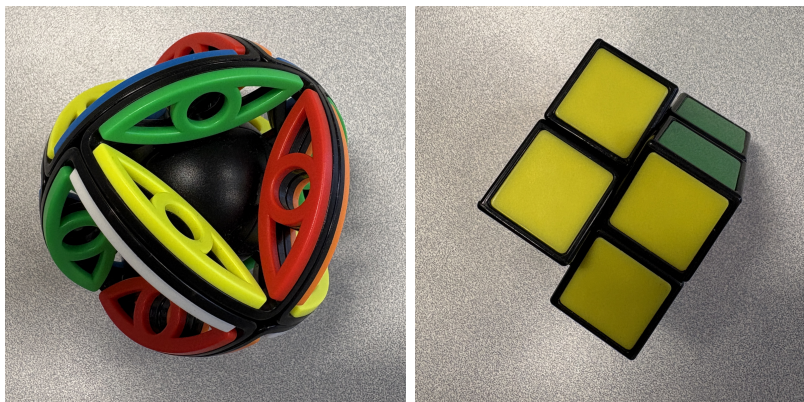


FIGURE 45. Two puzzles that are isomorphic.

to a solution path with a decent length. In the Rubik cube, the value system could be the number of correct cubes. In the chess case, it can be the sum of the values of each captured piece. In the 15 game it can be a distance between permutations. In a transposition puzzle, we have seen the graph distance  $d_H(x, y)$  on  $G$  which uses the distance matrix from the host graph  $\Gamma$  telling which transpositions can be used to bring the permutation  $x$  to the permutation  $y$ .

**13.8.** We should always look also whether backtracking is efficient. Start at the solution set  $A$  and take a few steps back to reveal part of the basin of attraction, positions which have to end up in  $A$ . This can be done in the case of the **peg solitaire**. If we can also analyze a neighborhood of  $v$ . If  $N_n(v) \cap N_m(A)$  intersect we might already see the god number.

**13.9.** One could also cover the entire graph by contractible graphs and build smaller graphs in which the contractible parts are identified. This is at the heart of the Hanoi puzzle for example. If we glue two disks together, we get from a Hanoi  $n$  puzzle to a Hanoi  $n - 1$  puzzle. More generally, if we have finitely many points  $P_1, \dots, P_n$  and radii  $r_i$  such that the neighborhood graphs of  $N(P_i, r_i)$  intersect, then we can estimate the god number.

## 14. HISTORY

**14.1.** The history of game theory starts earlier than von Neumann and Morgenstern [53]. A good source reference is [20]. James Waldegrave first described minimax solutions to best strategies in two-person zero sum card games. But as pointed out in the introduction to [20] Waldegrave did not attract attention so that the first formal reasoning for game theory are attributed to Cournot and Edgeworth 1897 who pioneered **oligopoly theory** to model industries with a few producers. An important early start in mathematical game theory with von Neumann proved in 1928 the **minimax theorem**. Von Neumann's **minimax theorem** tells that  $P^n$  is the space of probability vectors in  $\mathbb{R}^x$  and  $A$  is a  $n \times m$  matrix, then the bi-linear function  $f(x, y) = x \cdot Ay$  satisfies  $\max_{x \in P^n} \min_{y \in P^m} f(x, y) = \min_{y \in P^m} \max_{x \in P^n} f(x, y)$ . Von Neumann considered simplices like  $P^n$  **strategy sets** and  $f(x, y)$  the **expected payoff** when the first player plays  $x$  and the second player plays  $y$ . It latter got generalized to concave-convex functions, functions  $f(x, y)$  that are concave in  $x$  and convex in  $y$ .

**14.2.** If the theory of games is seen from a finitist point of view one can pretend to be able to oversee the entire game graph. In this “microcanonical perspective” probabilistic thinking is not really needed. When playing a traditional game like chess or checkers, we are in a finite setting and do in principle not need probability spaces. In a discrete finite frame work, in principle, we could do perfectly rational decisions by checking **all possible game outcomes**. While impractical

from a computer science point of view, mathematically speaking, we are on firm ground. When playing on a  $4 \times 4$  chess board with a select number of figures, we can draw out the entire game graph. Game graphs are usually drawn as trees (as for example in the textbook [58]). Indeed, if we look at the game graph  $G$  of a game, we can draw in principle build a tree where each vertices consist of **all possible simple paths starting in  $v$**  and two such paths are connected by an edge, if one is contained in the other and the length differ by 1. This is called the **self-avoiding walk tree** rooted at  $v$ . This SAW tree contains all the information about  $G$  but different graphs can produce the same tree.

**14.3.** Kalmar [38] already used a remarkable clarity. The 1928 article starts with: *“Ich werde mich mit solchen Spielen beschäftigen, welche von zwei Spielern durch abwechselnde Züge geführt werden, deren Wahl und Durchführbarkeit nur von dem Entschlusse des am Zug befindlichen Spielers bzw. von den Spielregeln, aber keineswegs vom Zufall oder von der Handfertigkeit des Spielers abhängt, und welche der Reihe nach Positionen herbeiführen, die beiden Spielern vollständig bekannt sind. (Spiele von der Art der Kartenspiele werden also durch diese Festsetzung ausgeschlossen). Das bekannteste Beispiel bietet das Schachspiel; andere Beispiele: das Damenspiel, Mühlenspiel, Nimspiel.)”* which translates to *“I shall concern myself with those games which are played by two players making alternating moves, whose choice and feasibility depend only on the decision of the player whose turn it is, or on the rules of the game, but in no way on chance or on the manual dexterity of the player, and which successively lead to positions that are completely known to both players. (Games of the kind of card games are therefore excluded by this stipulation.) The best-known example is chess; other examples are checkers, Nine Men’s Morris, and Nim.”* Note however that a card game like Poker that is excluded by Kalmar is still a finite situation. There is a game position for any possible permutation of the  $52!$  permutations of a card deck. The game graph is big, but still finite. Poker could be described by a finite game graph.

**14.4.** Von Neumann and Morgenstern used the idea of game tree already but the modern formal tree language first at about the same time, notably by Section 15.7 ”Application to Chess” of [53], the outcome of chess is encoded in a value function taking values  $1, 0, -1$ , where  $1$  means that  $V$  (White) has a strategy to win and  $-1$  means that  $W$  (Black) has a strategy to win and  $0$  means that both have a strategy such that the outcome is a tie. The mini-max principle comes in as this value is  $\max_x \min_y H(x, y) = \min_y \max_x H(x, y)$ . The  $x, y$  are however not probability vectors in this case. If we look at the axioms [53] (included in a figure below), this is hardly used when thinking about chess.

**14.5.** Even in [53] the notion of ”strategy” is rather vague when applied to chess. Seeing a strategy as a subgraph of the game graph is not only more intuitive it is purely combinatorial. To illustrate the language comparison, look at section 15.1.1 in [53]:  *$x, y$  in  $H(x, y)$  stand for ultima analysi not for a choice (in a move) but for the players’ strategies; i.e. their entire ”theory” or ”plan” concerning the game. For the moment we leave it at that. What does ”theory” or ”plan” mean? Defining ”strategies” as subgraphs of the game graph does not need to be explained.*

**14.6.** Non-deterministic features come in for two-player games already if both players have to make decisions at the same time like for a **prisoner dilemma** or **rock-paper-scissor game**, not knowing what the other is doing. An other source of randomness can enter in from external parameters, like the result of a **dice roll** like in Backgammon. In economic situations, it occurs also in complex setups, where many parameters can only be estimated. Also if the game graphs are too large, a deep search through all the possible game outcomes is impossible.

**14.7.** The mathematics of household games and puzzles which are most popular can be dealt with mathematics that is discrete: graph theory or combinatorial group theory. It might not be the most practical approach for usual games, especially if dice calls are invoked, where the game graph would have to include all the possible dice outcomes  $\{1, \dots, 6\}^n$  when considering moves of length  $n$ . In a card game like **black jack** we would have to look at the finite set of all  $(k * 52)!$  shufflings if a shoe of  $k$  card decks is used. Drawing out the entire **game graph** is obviously impractical already for much smaller card decks.

**14.8.** Zermelo's paper [81] is considered the first mathematical analysis of strategies in games. The notes of Zermelo (not even 4 pages long), were not very detailed and Denes Koenig fixed some gaps in 1927. Zermelo for example did not restrict to finite games. At that time, a "strategy" specified a specific move for each position meaning that one has not to take the maximum over all game events. A "quasi strategy" specifies an acceptable set of moves. The restriction to finite games dates back to von Neumann and Morgenstern from 1944. We can achieve that by assuming that a game position can not be reached twice.

**14.9.** The first popular group games was the 15 puzzle, a sliding puzzle that is modeled by the group  $S_{16}$  of all permutations of tiles in a  $4 \times 4$  board. A parity functional shows that the game group is the alternating group  $A_{16}$ . The most popular group game still today are Rubik type puzzles where the game graph is the Cayley graph of a group. The natural description is the language of finitely presented groups. As the groups are finite and every finite group is a permutation group we can restrict to permutation groups. From the book [14]: "until about 1850, the term 'group' referred to a set  $G$  of transformations of a set  $F$ , such that  $G$  is closed under composition, contains the identity transformation, and contains the inverse of each of its elements. This is what would now be called a 'permutation group'." [16].

**14.10.** For the axiomatic set up of Von Neumann and Morgenstern axiom system for  $n$ -player games, see section 10.1.1 in [53]. As mentioned before, the bulk of game theory deals with multiplayer games but this is difficult as different partitions of players must be considered. We cite Roth in [68]: *A theory of how players in a game might choose to organize themselves remains one of the most difficult and important problems in game theory.* Shapley [73] (who had studied at Harvard as an undergraduate) became the founder of **cooperative game theory**. In one of the most cited works in game theory thinks a "game" as a set of rules with specified players  $U$  in the playing positions and introduces then super additive valuation  $\nu$  from  $2^U$  to  $\mathbb{R}$  meaning  $\nu(A) \geq \nu(A \cap B) + \nu(A \setminus B)$  with the special additive case where  $\nu$  comes from a value  $\nu(A) = \sum_{i \in A} \phi(i)$  etc.

**14.11.** Obviously, economists look more at situations where games have lots of players. The analysis of "value functions" can then become important. Shapley and Nash knew each other as they were both young game theorists in Princeton. Nash developed "non-cooperative equilibrium theory", while Shapley focused on **cooperative games**. The umbrella was **Albert Tucker** who supervised Gale, Nash and Shapley and coined the term "**Prisoner Dilemma**" in 1950 and is sometimes called the "**midwife of game theory**". It is interesting that Tucker got his PhD under Solomon Lefschetz working on manifolds. Lefschetz is known among other things for his fixed point theorem generalizing the Brouwer fixed point theorem. Both of his academic grandsons Gale and Nash got involved in fixed point theorems again, Gale giving a spectacular proof of Brouwer's fixed point theorem using game theory and Nash giving an elegant proof of the von Neumann minimax theorem using the Brouwer fixed point theorem, and then developing the concept of Nash equilibria.

## 15. SOME CODE

**15.1.** We see first a simple 5 line implementation of the **game graph construction**. The input is a connected graph  $\Gamma$  with  $n$  vertices. The output is the **game graph graph**  $G$ , the Cayley graph generated by the transpositions generated by the edges of  $\Gamma$ . We have mentioned the folklore result that  $G$  is  $S_n$  and so has  $n!$  vertices. Our code is slightly slower than the **breadth first search construction** which builds up the balls  $B_r(x)$  layer by layer from a starting point  $x$ . We should note however that breadth first search does not necessarily get the graph diameter, as the center  $x$  of the ball has been chosen at random. While it is true for Cayley graphs that we always get the graph diameter with breadth first search, it is not true in general. The graph diameter is not always  $\min\{r, B_r(x) = G\}$ , where  $B_r(x)$  is the ball of radius  $r$ . The wheel graph  $G$  for example has diameter 2 but  $B_1(x) = G$  already with radius  $r = 1$ . The following construction it is almost as fast, as for connected graphs, the game graph has always  $S_n$  as vertex list. And most importantly, since we construct the game graph, we get its diameter for certain and not just a lower bound. We observe experimentally that all cases seen so far are Hamiltonian.

```

GameGraph[gamma.]:=Module[{e,v,A,B,K},
e=EdgeList[gamma]; e=Table[Sort[{e[[k,1]],e[[k,2]]}],{k,Length[e]};
v=VertexList[gamma]; n=Length[v]; A=Permutations[v]; K=Length[A];
T[a_,{p_,q_}]:=Table[If[k==p,a[[q]],If[k==q,a[[p]],a[[k]]]],{k,n};
B={};Do[a=A[[k]]; Do[b=T[a,e[[1]]]; B=Append[B,a->b],{1,Length[e]},{k,K}];
UndirectedGraph[Graph[B]]; TranspositionPuzzle=GameGraph;

S=GraphPlot3D[GameGraph[CompleteGraph[3]]]; Export["gamegraph_k3.pdf",S,"PDF"]
S=GraphPlot3D[GameGraph[PathGraph[Range[4]]]]; Export["gamegraph_p4.pdf",S,"PDF"]
S=GraphPlot3D[GameGraph[StarGraph[4]]]; Export["gamegraph_s4.pdf",S,"PDF"]
Table[GraphDiameter[GameGraph[PathGraph[Range[k]]]],{k,1,7}]
Table[GraphDiameter[GameGraph[CycleGraph[k]]],{k,1,7}]
Table[GraphDiameter[GameGraph[WheelGraph[k]]],{k,1,7}]
Table[GraphDiameter[GameGraph[StarGraph[k]]],{k,1,7}]
GraphDiameter[GameGraph[CompleteGraph[{2,2,2}]]] (* Gives 7, the same as GAP *)
Do[G=GameGraph[RandomGraph[{6,8}]]; Print[GraphDiameter[G]];
Print[HamiltonianGraphQ[G]},{100}]

(* the following takes hours, look at the BFS below which is faster *)
moebius=UndirectedGraph[Graph[{1->2,2->3,3->4,4->5,5->6,6->7,7->8,8->1,
1->5,5->2,2->6,6->3,3->7,7->4,4->8,8->5}]];
cylinder=UndirectedGraph[Graph[{1->2,2->3,3->4,4->1,5->6,6->7,7->8,8->5,
1->5,5->2,2->6,6->3,3->7,7->4,4->8,8->1}]];
GraphDiameter[GameGraph[moebius]]
GraphDiameter[GameGraph[cylinder]]

```

**15.2.** Here is some mathematica code which computes the lengths of the BFS layers  $|S_r(v)|$  (wave front layers) and compute it in the case of the Moebius strip and in the case of the Cylinder

```

T[a_,X_.]:=Module[{b=a}, b[[X[[1]]]]=a[[X[[2]]]]; b[[X[[2]]]]=a[[X[[1]]]]; b];
BFS[gamma.]:=Module[{e=EdgeList[gamma],v=VertexList[gamma],A,A0,A1,L={}},
n=Length[v]; A0={Range[n]}; A=A0; While[Length[A0]>0,
A1=Flatten[Table[T[A0[[k]],e[[1]]],{1,Length[e]},{k,Length[A0]}],1];
A1=Complement[A1,A]; A=Union[A,A1]; A0=A1; L=Append[L,Length[A1]]; L];
moebius =UndirectedGraph[Graph[{1->2,2->3,3->4,4->5,5->6,6->7,7->8,8->1,
1->5,5->2,2->6,6->3,3->7,7->4,4->8,8->5}]];
cylinder=UndirectedGraph[Graph[{1->2,2->3,3->4,4->1,5->6,6->7,7->8,8->5,
1->5,5->2,2->6,6->3,3->7,7->4,4->8,8->1}]];
{BFS[moebius],BFS[cylinder]}

```

**15.3.** The result  $BFS(moebius) = (0, 16, 137, 781, 3038, 7818, 12375, 10845, 4581, 700, 28, 0)$  and  $BFS(cylinder) = (0, 16, 136, 784, 3104, 8008, 12480, 10606, 4383, 744, 56, 2, 0)$  confirm that the diameter of the Cayley graphs are different. And here is the computation in GAP:

```

LoadPackage("grape"); # we use the moebius strip
a:=(1,2); b:=(2,3); c:=(3,4); d:=(4,5); e:=(5,6); f:=(6,7); g:=(7,8); h:=(8,1);
i:=(1,5); j:=(5,2); k:=(2,6); l:=(6,3); m:=(3,7); n:=(7,4); o:=(4,8); p:=(8,5);
G:= Group(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p);
S:=GeneratorsOfGroup(G); C:=CayleyGraph(G,S);; Diameter(C); # 10

a:=(1,2);b:=(2,3);c:=(3,4);d:=(4,1);e:=(5,6);f:=(6,7);g:=(7,8);h:=(8,5);
i:=(1,5);j:=(5,2);k:=(2,6);l:=(6,3);m:=(3,7);n:=(7,4);o:=(4,8);p:=(8,1);
G:= Group(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p);
S:=GeneratorsOfGroup(G); C:=CayleyGraph(G,S);; Diameter(C); # 11

```

**15.4.** The cycle graph case  $\Gamma = C_n$  is covered by the **theorem of Adin-Alon-Roichman**: [65] giving  $\text{God}(C_n) = \langle (n^2 + 1)/4 \rangle$ . We have given upper and lower bounds in the wheel case (in the section about transposition games) The path case is the **bubble sort case** with  $n(n - 1)/2$  diameter. Of course if  $\Gamma_1 \subset \Gamma_2$ , then the god number of  $\Gamma_1$  is larger or equal than the god number of  $\Gamma_2$ .

Size n	Complete	Path	Cycle	Wheel	Star
3	2	3	2	2	3
4	3	6	4	3	4
5	4	10	6	5	6
6	5	15	9	6	7
7	6	21	12	9	9
8	7	28	16	10	10
9	8	36	20	12	12
10	9	45	25	13	13
11	10	55	30	15	15

**15.5.** We implemented the algorithm both in Mathematica as well as in the GAP software [27] (both in OS X and linux). Both cases computed in Cayley graphs. Here is an example how to compute the god number. We take  $\Gamma = K_{2,2,2}$ , which is the **octahedron graph**, for which the god number is 7. The following GAP code confirms this. We used GAP also to compute the  $n=10, n=11$  wheel god numbers where it took already hours in the case of  $n = 11$ . Note that no formula is yet known for the wheel graph. It seems to agree with the star graph case for larger  $n$ . This means that sorting through a central node suffices and no sorting along the circular rim is needed. We have also Mathematica code for transposition and sliding puzzles defined by any graph  $\Gamma$  in the code section. For the **magic octahedron**, the case when we have a sliding puzzle with one hole, the god number is 7. Here is the GAP code for the **octahedron transposition puzzle**, giving graph diameter is 7.

```

LoadPackage("grape");
a:=(1,2); b:=(2,3); c:=(3,4); d:=(4,1); e:=(1,5); f:=(2,5);
g:=(3,5); h:=(4,5); i:=(1,6); j:=(2,6); k:=(3,6); l:=(4,6);
G:=Group(a,b,c,d,e,f,g,h,i,j,k,l);
S:=GeneratorsOfGroup(G);
C:=CayleyGraph(G,S);
Diameter(C);

```

**15.6.** Let us remark that if the diameter of the game graph is larger than 2, then the game graph  $G$  obtained from  $\Gamma$  with  $m$  edges one can try to produce a  $m$ -**manifold** by placing a  $m$ -simplex at every vertex of  $G$  and attach at every of the  $(m - 1)$  faces an other  $m$ -simplex. In the case of the path graph  $P_4$  with 4 vertices, it is an **Archimedian solid**, the **great rhom-bicuboctahedron** a graph with god number 6. Its dual is a **Catalan solid**, the **disdyakis dodecahedron** with 45 faces. It is a 2-sphere: every unit sphere  $S_1(x)$ , the boundary of  $B_1(x)$  is a circular graph. It is

not always the case that we get a manifold. An example of a 3-regular graph that is not the dual of a 2-manifold is the 4-Möbius ladder (Wagner graph).

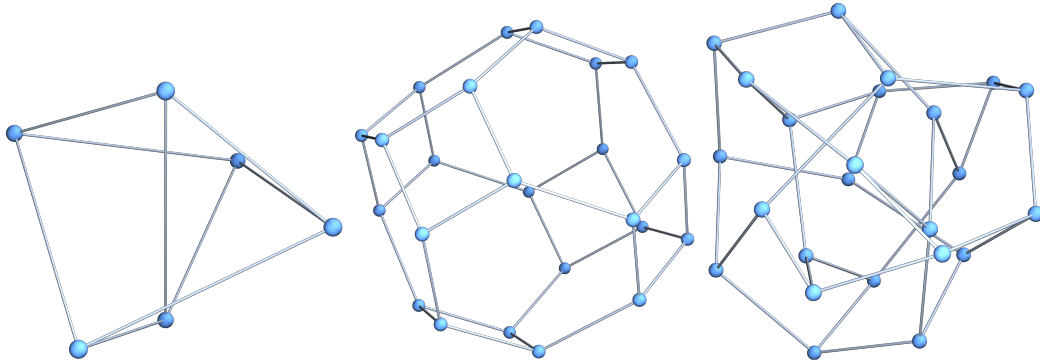


FIGURE 46. The game graph of  $K_3$  is the **utility graph**  $K_{3,3}$ . The game graph of the path graph  $P_4$  of length 3 is the dual of a 2-sphere (a Catalan solid in this case) The game graph of the star graph  $S_4$  is the dual of of a 2-manifold. The pictures were generated with the program given above.

**15.7.** Below is GAP code for the Rubik cube. It has been used in the 80ies. (It was used in student projects for a mathematical software course, where one of us (O.K) was an undergraduate course assistant [54]. The assignment given to the students in that "mathematical software" course was to write a program in CAYLEY (now MAGMA) to build a strategy that would solve the Rubik cube from any initial condition, build on a theory of Schreier-Sims.)<sup>11</sup> The students had to solve this homework assignment in the algebra programming language CAYLEY (which is syntax close to GAP today and now called MAGMA). The computer would randomly have to find generators of the stabilizer group, until a way is found to find a larger stabilizer group etc.

**15.8.** We compute here the god number of the subgroup of the Rubik cube, generated by face rotations by 180 degrees. This is again an example of a Cayley graph generated by involutions. But each generator is a product of two transpositions, not one. The god number is 15. The game graph has 663'552 vertices and 1'990'656 edges, girth 4.

As for an other example, the group generated by three generators  $a^2, b^2, c^2$  has only 2592 vertices and 3888 edges, girth 12 and diameter 14. The group Rubik itself produces a game graph 43252003274489856000 that is too large for the machine to get the diameter for. It is 20, if one uses the generators  $a^k, b^k, \dots, f^k$  for  $k = 1, 2, 3$ .

```

LoadPackage("grape");
a:=( 1, 3, 8, 6)( 2, 5, 7, 4)( 9,33,25,17)(10,34,26,18)(11,35,27,19);
b:=( 9,11,16,14)(10,13,15,12)( 1,17,41,40)( 4,20,44,37)( 6,22,46,35);
c:=(17,19,24,22)(18,21,23,20)( 6,25,43,16)( 7,28,42,13)( 8,30,41,11);
d:=(25,27,32,30)(26,29,31,28)( 3,38,43,19)( 5,36,45,21)( 8,33,48,24);
e:=(33,35,40,38)(34,37,39,36)( 3, 9,46,32)( 2,12,47,29)( 1,14,48,27);
f:=(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40);
Rubik := Group(a,b,c,d,e,f);
Size(Rubik);
G:=Group(a^2,b^2,c^2,d^2,e^2,f^2);
S:=GeneratorsOfGroup(G);

```

<sup>11</sup>When kids got the Rubik cube in the late 1970'ies, the idea of the game was to build your own strategy. For O.K., this had required weeks during high school. When looking and comparing such algorithms later, it became clear that everybody (without knowing any group theory) developed independently some sort of **Schreier-Sims algorithm**, working in a tower of **stabilizer groups**, fixing larger and larger parts of the cube, then developing generators for that group. It is interesting that kids are often better than adults in that, maybe because they are not afraid to play around, even if they do not know what they are doing.

```
C:=CayleyGraph(G,S);
Size(UndirectedEdges(C));
Size(G);
Girth(C);
Diameter(C);
```

**15.9.** Here is the GAP program for the floppy: [27]

```
# Gap code for Floppy group, God number 13, 2304 elements
#
#           +-----+
#           | 1   2   3 |
#           | 4 top 5 |
#           | 6   7   8 |
# +-----+-----+-----+-----+
# | 9  10  11 | 12  13  14 | 15  16  17 | 18  19  20 |
# +-----+-----+-----+-----+
#           | 21  22  23 |
#           | 24 bottom 25 |
#           | 26  27  28 |
#           +-----+
LoadPackage("grape");
a:=(1,28)(2,22)(3,26)(18,20);
b:=(6,23)(8,21)(7,22)(12,14);
c:=(1,26)(6,21)(4,24)(12,20);
d:=(3,28)(8,23)(5,25)(14,18);

G:=Group(a,b,c,d);
S:=GeneratorsOfGroup(G);
C:=CayleyGraph(G,S);
Diameter(C);
Size(G);
```

**15.10.** Here is a Mathematica implementation of the floppy with one cube fixed (space symmetries are factored out):

```
(* Floppy Rubik Cube implementation: from Oliver Knill 2011 for Math E 320 *)
T1[{x-,y-}]:={{x[[2]],x[[1]],x[[3]],x[[4]]},{-y[[1]],-y[[2]],y[[3]],y[[4]]}};
T2[{x-,y-}]:={{x[[1]],x[[3]],x[[2]],x[[4]]},{y[[1]],-y[[2]],-y[[3]],y[[4]]}};
T3[{x-,y-}]:={{x[[1]],x[[2]],x[[4]],x[[3]]},{y[[1]],y[[2]],-y[[3]],-y[[4]]}};
T4[{x-,y-}]:={{x[[4]],x[[2]],x[[3]],x[[1]]},{-y[[1]],y[[2]],y[[3]],-y[[4]]}};
T5[{x-,y-}]:={{x[[3]],x[[2]],x[[1]],x[[4]]},{-y[[1]],y[[2]],-y[[3]],y[[4]]}};
T6[{x-,y-}]:={{x[[1]],x[[4]],x[[3]],x[[2]]},{y[[1]],-y[[2]],y[[3]],-y[[4]]}};

P=Permutations[{1,2,3,4}]; T={T1,T2,T3,T4,T5,T6};
s=Partition[Flatten[Table[{a,b,c,a*b*c},{a,-1,1,2},{b,-1,1,2},{c,-1,1,2}],4];
v=Partition[Flatten[Table[{P[[k]],s[[1]]},{k,Length[P]},{1,Length[s]}],8];
v=Table[Partition[v[[k]],4],{k,Length[v]}];
e=Flatten[Table[Table[v[[k]]->T[[1]][v[[k]]],{k,Length[v]},{1,Length[T]}];
s=UndirectedGraph[Graph[e]]; S=GraphPlot3D[s];
Print["Is-Hamiltonian: -",HamiltonianGraphQ[s]];
Print["God-Number: ---",GraphDiameter[s]]
```

**15.11.** Here is the gap code for the  $2 \times 2 \times 2$  Rubik (pocket cube) which produces a graph with  $8! \cdot 3^7 = 88179840$  vertices. The diameter of the Cayley is 14. The code needed a couple of hours to run.

```
LoadPackage("grape");
a:=(1,2,3,4)(5,17,13,9)(6,18,14,10);
b:=(5,6,7,8)(1,9,21,19)(4,12,24,18);
c:=(13,14,15,16)(2,20,22,10)(3,17,23,11);
d:=(9,10,11,12)(4,13,22,7)(3,16,21,6);
e:=(17,18,19,20)(1,8,23,14)(2,5,24,15);
f:=(21,22,23,24)(11,15,19,7)(12,16,20,8);
G:= Group(a,b,c,d,e,f);
S:=GeneratorsOfGroup(G);
```

```
C:=CayleyGraph(G,S);
Diameter(C);
Size(G);
```

**15.12.** And here is a GAP implementation of the masterball. It had been advertised as a challenging puzzle because it has more positions than Rubik. But it is actually easier as a,b,c,d commute and the second and third layers can not mix with the first and fourth reflecting that the group is  $S_{16} \times S_{16}$  a subgroup of  $S_{32}$ .<sup>12</sup>

```
LoadPackage("grape");
a:=(1 , 2, 3, 4, 5, 6, 7, 8 );
b:=(9 ,10,11,12,13, 14, 15, 16 );
c:=(17,18,19,20,21, 22, 23, 24 );
d:=(25,26,27,28,29, 30, 31, 32 );
e:=(1,25)(9,17)(2,26)(10,18)(3,27)(11,19)(4,28)(12,20);
G:=Group(a,b,c,d,e);
S:=GeneratorsOfGroup(G);
C:=CayleyGraph(G,S);
Diameter(C);
Size(G);
```

**15.13.** The group above is the product of two groups of size  $16! = 20922789888000$ . The Cayley graph has diameter 9.<sup>13</sup>

```
LoadPackage("grape");
a:=(1 , 2, 3, 4, 5, 6, 7, 8 );
b:=(9 ,10,11,12,13, 14, 15, 16 );
c:=(1,9)(2,10)(3,11)(4,12);
G:=Group(a,b,c);
S:=GeneratorsOfGroup(G);
C:=CayleyGraph(G,S);
Diameter(C);
Size(G);
```

**15.14.** The situation is interesting as small subgroups can be understood better. If one allows only rotations by 90 degrees one works in the smaller group of  $8!$  elements.

```
LoadPackage("grape");
a:=(1 , 2, 3, 4);
b:=(5 , 6, 7, 8);
c:=(1,5)(2,6);
G:=Group(a,b,c);
S:=GeneratorsOfGroup(G);
C:=CayleyGraph(G,S);
Diameter(C);
Size(G);
```

<sup>12</sup>masterball

<sup>13</sup>When the rainbow master ball was released during the 1980ies, the company organized a contest in the Swiss town of Bern, where the puzzle had to be solved competitively in front of a larger audience. The task was to perform a specific flip in the puzzle. O.K. used then Cayley (now Magma) on a Sun Workstation to come up with a path that would do the flip. The winner of the competition and winner to a trip to the US was a Swiss farmer and cheese-maker from Emmental. He had the best understanding to walk around in that Cayley graph, without computer!

**15.15.** Here is a Mathematica implementation for  $n \times m$  sliding puzzles:

```
SlidingPuzzleGrid[n_, m_] := Module[{}, R = Range[n * m]; R[[n * m]] = x;
f[X_] := Partition[X, m]; A = Map[f, Permutations[R]]; K = Length[A];
T[a_, b_] := Module[{p, q}, p = Position[a, x]; q = Position[b, x];
  Total[Flatten[p - q] == 1 && Delete[a, p] == Delete[b, q]];
B = {}; Do[a = A[[k]]; b = A[[1]]; If[T[a, b] ||
  T[Transpose[a], Transpose[b]], B = Append[B, a -> b]], {k, K}, {1, K}];
G = UndirectedGraph[Graph[A, B]];
G = SlidingPuzzleGrid[2, 3]; (* 5-puzzle, |V|=720, |E|=840 Edges. The god number is 21 *)
Print[Map[GraphDiameter, ConnectedGraphComponents[G]]];
H1 = First[ConnectedGraphComponents[G]]
```

**15.16.** We also programmed the sliding puzzle for a general graph  $\Gamma$ . In this case, the “hole” is the vertex with label 1. Unfortunately, for the **rainbow puzzle**, the case where  $\Gamma$  is the icosahedron graph, the program is not strong enough yet to get us the god number. As sliding puzzles are not Cayley graphs, we could also not just feed it to GAP.

```
SlidingPuzzleGraph[gamma_] := Module[{e, v, A, B, K, r, s},
e = EdgeList[gamma]; e = Table[Sort[{e[[k, 1]], e[[k, 2]]}], {k, Length[e]}];
v = VertexList[gamma]; n = Length[v]; A = Permutations[v]; K = Length[A];
T[a_, {p_, q_}] := Table[If[k == p, a[[q]], If[k == q, a[[p]], a[[k]]]], {k, n}];
B = {}; Do[a = A[[k]]; Do[{r, s} = e[[1]];
  If[a[[r]] == 1 || a[[s]] == 1, b = T[a, e[[1]]]; B = Append[B, a -> b]], {1, Length[e]}, {k, K}];
UndirectedGraph[Graph[B]];

gamma = CompleteGraph[{2, 2, 2}]; G = SlidingPuzzleGraph[gamma];
GraphDiameter[First[ConnectedGraphComponents[G]]] (* 11 *)
gamma = GridGraph[{2, 2}]; G = SlidingPuzzleGraph[gamma];
GraphDiameter[First[ConnectedGraphComponents[G]]] (* 6 *)
gamma = GridGraph[{3, 2}]; G = SlidingPuzzleGraph[gamma];
GraphDiameter[First[ConnectedGraphComponents[G]]] (* 21 *)
gamma = PolyhedronData["Icosahedron", "Skeleton"];
G = SlidingPuzzleGraph[gamma] (* too large *)
```

**15.17.** Here is Mathematica code to compute the Hanoi graphs:

```
check[{X_, Y_, Z_}] := (X == Sort[X] && Y == Sort[Y] && Z == Sort[Z]);
T12[{X_, Y_, Z_}] := If[Length[X] == 0, {X, Y, Z}, {Delete[X, 1], Prepend[Y, X[[1]]], Z}];
T13[{X_, Y_, Z_}] := If[Length[X] == 0, {X, Y, Z}, {Delete[X, 1], Y, Prepend[Z, X[[1]]]};
T21[{X_, Y_, Z_}] := If[Length[Y] == 0, {X, Y, Z}, {Prepend[X, Y[[1]]], Delete[Y, 1], Z}];
T23[{X_, Y_, Z_}] := If[Length[Y] == 0, {X, Y, Z}, {X, Delete[Y, 1], Prepend[Z, Y[[1]]]};
T31[{X_, Y_, Z_}] := If[Length[Z] == 0, {X, Y, Z}, {Prepend[X, Z[[1]]], Y, Delete[Z, 1]};
T32[{X_, Y_, Z_}] := If[Length[Z] == 0, {X, Y, Z}, {X, Prepend[Y, Z[[1]]], Delete[Z, 1]};
extend[x_] := Module[{},
y = T12[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];
y = T13[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];
y = T21[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];
y = T23[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];
y = T31[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];
y = T32[x]; If[check[y] && Not[x == y], v = Union[v, {y}]; e = Append[e, x -> y];];
extendgraph := Module[{}, Do[extend[v[[i]]], {i, Length[v]}];

HanoiGraph[n_] := Module[{}, x0 = {Range[n], {}, {}}; v = {x0}; e = {};
Do[extendgraph, 2^n]; s = UndirectedGraph[Graph[e]];
Print[{n, Length[VertexList[s]], Length[EdgeList[s]]};
Print[GraphDiameter[s]]; s];
s = HanoiGraph[4]
```

**15.18.** Here is the code which generates the game graph of the peg solitaire on a  $3 \times 3$  lattice. There is a component with maximal diameter 5. The component with one missing peg at the boundary has diameter 4.

GRAPHS,GAMES,GROUPS

```

n=3;A=Tuples[{0,1},n^2];p[x_]:=Partition[x,n];A=Map[p,A];
V[x_,y_]:=x[[1]]==y[[1]]&& x[[2]]==y[[2]]&&
((x[[3]]=={1,1,0} && y[[3]]=={0,0,1}) ||
(x[[3]]=={0,1,1} && y[[3]]=={1,0,0}));
X[x_,y_]:=V[x,y]||V[RotateRight[x,1],RotateRight[y,1]] ||
V[RotateRight[x,2],RotateRight[y,2]];
Y[x_,y_]:=X[Transpose[x],Transpose[y]];W[x_,y_]:=X[x,y]||Y[x,y];
B={}; Do[If[W[A[[k]],A[[1]]],
B=Append[B,A[[k]]->A[[1]]],{k,Length[A]},{1,Length[A]}];
G=UndirectedGraph[Graph[A,B]]
G1 = First[ConnectedGraphComponents[G]]; GraphDiameter[G1]; (* 5 *)
s=NeighborhoodGraph[G, A[[Length[A]-1]], 5];
S=GraphPlot[s, GraphStyle->"SmallNetwork"]

```

15.19. Here is code we used to illustrate the  $2 \times 4$  chess game with kings and queens only. It is highly inefficient but self-contained.

```

B={K,Q}, {0,0}, {0,0}, {k,q}; A=Reverse[Transpose[Reverse[B]]]; qm="q"; km="k"; Q="Q"; K="K";
R=MemberQ; P=Append;

WhiteMoves[A_]:=Module[{G={}},
(* White moves without catching *)
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i,j+1]]==0,F[[i,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,3}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i,j-1]]==0,F[[i,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,2,4}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i+1,j]]==0,F[[i+1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,4}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i-1,j]]==0,F[[i-1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,4}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i-1,j+1]]==0,F[[i-1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,3}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i-1,j-1]]==0,F[[i-1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,2,4}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i+1,j+1]]==0,F[[i+1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{q,k},A[[i,j]]]&&A[[i+1,j-1]]==0,F[[i+1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
(* White queen moves without catching *)
Do[{F=A; If[A[[i,j]]==q&&A[[i,j+1]]==A[[i,j+2]]==0,F[[i,j+2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,2}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j-1]]==A[[i,j-2]]==0,F[[i,j-2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,3,4}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j+1]]==A[[i,j+2]]==A[[i,j+3]]==0,F[[i,j+3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,1}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j-1]]==A[[i,j-2]]==A[[i,j-3]]==0,F[[i,j-3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,4,4}];
(* White king or queen catches straight *)
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i+1,j]]],F[[i+1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,4}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i-1,j]]],F[[i-1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,4}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i,j+1]]],F[[i,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i,j-1]]],F[[i,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
(* White king or queen catches diagonally *)
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i+1,j+1]]],F[[i+1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i+1,j-1]]],F[[i+1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i-1,j+1]]],F[[i-1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,3}];
Do[{F=A; If[R[{k,q},A[[i,j]]]&&R[{Q,K},A[[i-1,j-1]]],F[[i-1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,2,4}];
(* White more queen catches *)
Do[{F=A; If[A[[i,j]]==q&&A[[i,j+1]]==0&&R[{Q,K},A[[i,j+2]]],F[[i,j+2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,2}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j-1]]==0&&R[{Q,K},A[[i,j-2]]],F[[i,j-2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,3,4}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j+1]]==0&&A[[i,j+2]]==0&&R[{Q,K},A[[i,j+3]]],F[[i,j+3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,2},{j,1,1}];
Do[{F=A; If[A[[i,j]]==q&&A[[i,j-1]]==0&&A[[i,j-2]]==0&&R[{Q,K},A[[i,j-3]]],F[[i,j-3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,2},{j,4,4}];
G};

BlackMoves[A_]:=Module[{G={}},
(* Black moves without catching *)
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i,j+1]]==0,F[[i,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,3}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i,j-1]]==0,F[[i,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,2,4}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i+1,j]]==0,F[[i+1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,4}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i-1,j]]==0,F[[i-1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,4}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i-1,j+1]]==0,F[[i-1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,3}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i-1,j-1]]==0,F[[i-1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,2,4}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i+1,j+1]]==0,F[[i+1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i+1,j-1]]==0,F[[i+1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i-1,j+1]]==0,F[[i-1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,3}];
Do[{F=A; If[R[{Q,K},A[[i,j]]]&&A[[i-1,j-1]]==0,F[[i-1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,2,4}];
(* Black king or queen catches straight *)
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i+1,j]]],F[[i+1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,4}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i-1,j]]],F[[i-1,j]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,4}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i,j+1]]],F[[i,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i,j-1]]],F[[i,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
(* Black king or queen catches diagonally *)
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i+1,j+1]]],F[[i+1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,1,3}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i+1,j-1]]],F[[i+1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,1,1},{j,2,4}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i-1,j+1]]],F[[i-1,j+1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,1,3}];
Do[{F=A; If[R[{K,Q},A[[i,j]]]&&R[{q,k},A[[i-1,j-1]]],F[[i-1,j-1]]=A[[i,j]];F[[i,j]]=0; G=P[G,A->F]},{i,2,2},{j,2,4}];
(* Black more queen catches *)
Do[{F=A; If[A[[i,j]]==Q&&A[[i,j+1]]==0&&R[{q,k},A[[i,j+2]]],F[[i,j+2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,1,2}];
Do[{F=A; If[A[[i,j]]==Q&&A[[i,j-1]]==0&&R[{q,k},A[[i,j-2]]],F[[i,j-2]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,1,2},{j,3,4}];
Do[{F=A; If[A[[i,j]]==Q&&A[[i,j+1]]==0&&A[[i,j+2]]==0&&R[{q,k},A[[i,j+3]]],F[[i,j+3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,2},{j,1,1}];
Do[{F=A; If[A[[i,j]]==Q&&A[[i,j-1]]==0&&A[[i,j-2]]==0&&R[{q,k},A[[i,j-3]]],F[[i,j-3]]=q;F[[i,j]]=0; G=P[G,A->F]},{i,2},{j,4,4}];
G};

SetOptions[GraphPlot, BaseStyle -> {FontFamily -> "Times", FontSize -> 6}];
Gall={}; Vold={A}; G=WhiteMoves[A]; Gall=Union[Gall,G]; s=UndirectedGraph[Graph[G];
Vnew=VertexList[s]; V=Complement[Vnew,Vold]; A1=A; i0=1; j0=1; A1[[i0+1,j0+1]]=A[[i0,j0]]; A1[[i0,j0]]=0;
Vold=Vnew; Do[If[V[[k]]!=A1,G=BlackMoves[V[[k]]]; Gall=Union[Gall,G]; s=UndirectedGraph[Graph[Gall]];
Vnew=Union[Vnew,VertexList[s]]; {k,Length[V]}; V=Complement[Vnew,Vold];
S=GraphPlot[s, GraphStyle -> "SmallNetwork", GraphLayout -> "RadialDrawing"];

```

15.20. And here is the BFS layer computation in the example where the layer sequence is not concave down.

```

n=713; a=48; b=88; c=138; v=Range[n]-1; e={}; R={a,b,c,-a,-b,-c};
e=Flatten[Table[Table[v[[k]]->Mod[v[[k]]+R[[1]],n],{1,Length[R]}],{k,Length[v]}];
s=UndirectedGraph[Graph[e]]; v0=v[[1]]; d=GraphDiameter[s]; Print[d]; (* 10 *)
A=Table[Length[VertexList[NeighborhoodGraph[s,v0,k]]],{k,0,d}]; A=Prepend[A,0];
diff[X_]:=Table[X[[k+1]]-X[[k]],{k,Length[X]-1}; Print[diff[A]];

```

16. SUMMARY OF AXIOMS

16.1. Here is a summary of the axiomatic setup for 1-player games = solitaire games.

- 1) The **game graph**  $G$  of a **solitaire game** is a finite simple directed graph  $(V, E)$ .
- 2) A distinguished vertex  $v \in V$  is the **initial condition** of the game.
- 3) In the set  $S = \{x \in V, \text{outdeg}_G(x) = 0\}$  of **end positions**, we distinguish a set  $A \subset S$  of **win positions**.
- 4) A **strategy** is a sub-graph  $H$  of  $G$ . Let  $\mathcal{H}$  be the set of strategies.
- 5) A **game event**  $\gamma = (v = v_0, v_1, \dots, v_k)$  in  $H$  is a maximal simple path in  $H$  starting in  $v$ . Call them  $\mathcal{E}(H)$ .
- 6) The length  $|\gamma|$  of a game event is the number of edges in  $\gamma$ .
- 7)  $G$  is **winnable** if there exists  $H \in \mathcal{H}$  such that all game events in  $H$  end in  $A$ .
- 8) The **god number** of a winnable game is  $\text{God}(G, v, A) = d(v, A) = \min_{a \in A} d(v, a)$ .
- 9) The **maximal play length** of a non-winnable game is  $\max_{\gamma \in \mathcal{E}(G)} |\gamma|$ . Define  $\text{God}(G, v, A) = \infty$  then.

16.2. Without specifying  $v, A$  the corresponding global invariant is  $\max_{x,y \in V, y \text{ reachable from } x} d(x, y)$ . The god number of the graph, its diameter is an upper bound for the god number of the game (which has additional structure in the form of initial condition  $v$  and win positions  $A$ ). The diameter of the game graph can be larger than the god number of the game as the starting position  $v$  can be close to the solution set  $A$ . If  $v \in A$ , then  $\text{God}(G) = 0$ . The set of game events consists now of a single path of length  $|\gamma| = 0$ . In non-winnable cases, the god number is usually not evaluated. It is the maximal length of a simple path starting at  $v$ . For a non-winnable game, the maximal play length is  $(n - 1)$  if and only if  $G$  is **Hamiltonian** meaning that  $G$  admits a **Hamiltonian path**, a simple path covering all vertices. The god number computation leads in general to **NP complete problems** for solvable solitaire games, as the  $n \times m$  generalization of the 15-puzzle shows.

16.3. Here is the setup for 2-player games: for **normal play**. In **misère play**, the player who moves last loses, and  $A_V = A \cap V$  would be the win positions. In that case the last move would just not be done and  $A_W = A \cap W$ . But misere play could easily be transformed to normal play.

- 1) The **game graph** is a finite simple bipartite directed graph  $G = (V \cup W, E)$ . Vertices  $V \cup W$  represent possible game positions.
- 2) Player  $V$  plays from  $V$  to  $W$  and player  $W$  plays from  $W$  to  $V$  meaning  $E = E_V \cup E_W$  with  $E_V \subset V \times W$  and  $E_W \subset W \times V$ .
- 3) Denote by  $S = \{x \in V \cup W, \text{outdeg}_G(x) = 0\}$  the set of **end positions**.
- 4) A distinguished vertex  $v \in V$  is the **initial condition** of the game.  $v \in W$  can be arranged by switching labels.
- 5) There is a **win set**  $A \subset S$ . Terminal positions in  $A \cap W$  are wins for  $V$  and terminal positions in  $A \cap V$  are wins for  $W$ .
- 6) A **game event**  $\gamma$  in  $H \subset G$  is a maximal simple path in  $H$  that starts at  $v$ .
- 7) The **length** of a game event is defined as  $|\gamma| = |E_V \cap E(\gamma)| = |\{0 \leq j < k : v_j \in V, (v_j, v_{j+1}) \in E\}|$ . The length is measured in moves of the starting player  $V$ , regardless who eventually wins.
- 8) A **strategy** is a sub-graph  $H$  of  $G$  containing  $v$ . The strategy  $H = \{v\}$  would mean, to not play.
- 9) A **V-strategy** is a strategy  $H$  with  $E_W \subset E(H)$  meaning player  $V$  can limit moves of  $W$  but not of  $V$ .
- 10) A **W-strategy** is a strategy  $H$  with  $E_V \subset E(H)$  meaning player  $W$  can limit moves of  $V$  but not of  $W$ .
- 11) **G is a win for V**, if there is a V-strategy  $H$  such that every game event in  $H$  ends in  $A \cap W$ .
- 12) **G is a win for W**, if there is a W-strategy  $H$  such that every game event in  $H$  ends in  $A \cap V$ .
- 13) **G is a draw** if neither  $V$  nor  $W$  has a winning strategy.
- 14) If  $G$  is a win for  $V$ , the **god number** is  $\text{God}_V(G, v) = \min_{H \in \mathcal{H}_V^{\text{win}}} \max_{\gamma \in \mathcal{E}(H)} |\gamma|$ .
- 15) If  $G$  is a win for  $W$ , the **god number** is  $\text{God}_W(G, v) = \min_{H \in \mathcal{H}_W^{\text{win}}} \max_{\gamma \in \mathcal{E}(H)} |\gamma|$ .
- 16) If  $G$  is a draw, define  $\text{God}(G, v, A) = \infty$  and define maximal play length  $L_{\max}(G, v) = \max_{\gamma \in \mathcal{E}(G)} |\gamma|$ .

**16.4. Zermelo’s theorem** logically follows from the axioms: **a game is either a win for V, a win for W or a draw**. Since a win for  $V$  excludes a win for  $W$  the fourth logical possibility that both do win can not happen. Indeed, if  $H_V$  is a winning  $V$ -strategy and  $H_W$  is a winning  $W$ -strategy, then their intersections would be a strategy such that only game events that end both in  $A \cap W$  as well as  $A \cap V$ , which is impossible due to the bipartite structure implying  $V$  and  $W$  being disjoint. One can define the combined play strategy  $H = H_V \cap H_W$ . Any maximal event in  $H$  would be forced to go into two disjoint sets  $A \cap V$  and  $A \cap W$ . **Chess problems** “mate in  $k$ ” (with a slightly modified game rule so that a **repeated position immediately lead to a draw**) has god number  $k$ . We have assumed  $v \in V$ . In a chess problem “Black to move and mate in 3”, player Black is  $V$ . Note that  $|\gamma|$  always counts the number of moves of the first player  $V$ .

**16.5.** There are various statements which can be deduced immediately from the axioms. Examples:

1) The set of strategies forms a  $\pi$ -system. This means that it a set of objects that is closed under intersection. If  $H_1, H_2$  are strategies, then  $H_1 \cap H_2$  are a strategy. Also V-strategies and W-strategies form  $\pi$  systems.

2) The set of game events in  $H$  naturally form a new graph that is a tree  $T(H)$ . This is the **self-avoiding walk tree** of  $H$  rooted at  $v$ . This tree  $T(H)$  allows to reconstruct  $H$ , but do not necessarily characterize  $H$  as there could be different trees that represent the same graph.

3) The length function  $|\gamma|$  counting the number of moves of the first player defines a distance notion  $d(v, y) = \inf_{\gamma} |\gamma|$  of points to the initial condition  $v$ . It is not a metric because  $d(y, v)$  is not defined as  $\gamma$  runs over all game events starting at  $v$  and ending at  $y$ . One can assume  $d(v, y) = \infty$  if there is no game path from  $v$  to  $y$ . Computer scientists usually look at balls  $B_r(v)$  in BFS search and look only at  $G' = B_{\text{God}(G)}(v) \subset G$  as these are all the reachable configurations. As this sub-graph of  $G$  is again a game graph, we could just restrict  $G'$  and get the new equilibrium set  $S' = S \cap G'$  and win set  $A' = A \cap G'$ .

4) An other possibility is to shift the initial condition  $v$  to an other starting point  $w$  and look at the game  $G' = \bigcup_r B_r(w)$  with initial condition  $v' = w$  and winning set  $A' = A \cap G'$  and equilibrium set  $S' = S \cap G'$ . An example is to look at an initial chess position, where only a few figures are present. A god number  $k$  for a win of  $V$  means then  $\text{God} = k$ .

**16.6.** We see with such examples that the setup is close to **down to earth game situations** encountered when playing traditional games or solving traditional puzzles. In order to appreciate that, one has to look and compare with other axiomatic setups. The above graph theoretical axioms are **down to earth axioms**.

## 10. Axiomatic Formulation

### 10.1. The Axioms and Their Interpretations

**10.1.1.** Our description of the general concept of a game, with the new technique involving the use of sets and of partitions, is now complete. All constructions and definitions have been sufficiently explained in the past sections, and we can therefore proceed to a rigorous axiomatic definition of a game. This is, of course, only a concise restatement of the things which we discussed more broadly in the preceding sections.

We give first the precise definition, without any commentary:<sup>2</sup>

An  $n$ -person game  $\Gamma$ , i.e. the complete system of its rules, is determined by the specification of the following data:

- (10:A:a)            A number  $\nu$ .
- (10:A:b)            A finite set  $\Omega$ .
- (10:A:c)            For every  $k = 1, \dots, n$ : A function  
 $\mathfrak{F}_k = \mathfrak{F}_k(\pi), \quad \pi \text{ in } \Omega$ .
- (10:A:d)            For every  $\kappa = 1, \dots, \nu, \nu + 1$ : A partition  $\mathfrak{G}_\kappa$  in  $\Omega$ .
- (10:A:e)            For every  $\kappa = 1, \dots, \nu$ : A partition  $\mathfrak{B}_\kappa$  in  $\Omega$ .  $\mathfrak{B}_\kappa$  consists of  $n + 1$  sets  $B_\kappa(k), k = 0, 1, \dots, n$ , enumerated in this way.
- (10:A:f)            For every  $\kappa = 1, \dots, \nu$  and every  $k = 0, 1, \dots, n$ : A partition  $\mathfrak{C}_\kappa(k)$  in  $B_\kappa(k)$ .
- (10:A:g)            For every  $\kappa = 1, \dots, \nu$  and every  $k = 1, \dots, n$ : A partition  $\mathfrak{D}_\kappa(k)$  in  $B_\kappa(k)$ .
- (10:A:h)            For every  $\kappa = 1, \dots, \nu$  and every  $C_\kappa$  of  $\mathfrak{C}_\kappa(0)$ : A number  $p_\kappa(C_\kappa)$ .

These entities must satisfy the following requirements:

- (10:1:c)            For  $k = 1, \dots, n$ :  $\mathfrak{C}_\kappa(k)$  is a subpartition of  $\mathfrak{D}_\kappa(k)$ .
- (10:1:d)            For  $k = 1, \dots, n$ : Within  $B_\kappa(k)$ ,  $\mathfrak{G}_\kappa$  is a subpartition of  $\mathfrak{D}_\kappa(k)$ .
- (10:1:e)            For every  $\kappa = 1, \dots, \nu$  and every  $A_\kappa$  of  $\mathfrak{G}_\kappa$  which is a subset of  $B_\kappa(0)$ : For all  $C_\kappa$  of  $\mathfrak{C}_\kappa(0)$  which are subsets of this  $A_\kappa, p_\kappa(C_\kappa) \geq 0$ , and for the sum extended over them  $\Sigma p_\kappa(C_\kappa) = 1$ .
- (10:1:f)             $\mathfrak{G}_1$  consists of the one set  $\Omega$ .
- (10:1:g)             $\mathfrak{G}_{\nu+1}$  consists of one-element sets.
- (10:1:h)            For  $\kappa = 1, \dots, \nu$ :  $\mathfrak{G}_{\kappa+1}$  obtains from  $\mathfrak{G}_\kappa$  by superposing it with all  $\mathfrak{C}_\kappa(k), k = 0, 1, \dots, n$ . (For details, cf. 9.2.2.)
- (10:1:i)            For  $\kappa = 1, \dots, \nu$ : If  $A_\kappa$  of  $\mathfrak{G}_\kappa$  and  $C_\kappa$  of  $\mathfrak{C}_\kappa(k), k = 1, \dots, n$  are subsets of the same  $D_\kappa$  of  $\mathfrak{D}_\kappa(k)$ , then the intersection  $A_\kappa \cap C_\kappa$  must not be empty.
- (10:1:j)            For  $\kappa = 1, \dots, \nu$  and  $k = 1, \dots, n$  and every  $D_\kappa$  of  $\mathfrak{D}_\kappa(k)$ : Some  $C_\kappa(k)$  of  $\mathfrak{C}_\kappa$ , which is a subset of  $D_\kappa$ , must exist.

FIGURE 47. A page of [53] listing the axioms.

game, what his choices for all these steps are going to be. It would mean a restriction of his free will and change his chances (for the worse). For,  $S_m$ 's decision in each of these "steps" will generally be significantly influenced by the results of the "draws" and "steps" known to him at the moment of his decision.

On the other hand, it may well be assumed that before the play has started he knows how to answer the following question: What will be the outcome of the  $v_k^{(m)}$ -th "step" ( $k = 1, 2, \dots, \sigma_m$ ) provided the results of all "draws" and "steps" "earlier" than  $v_k^{(m)}$  are available? In other words, the player knows beforehand how he is going to act in a precisely defined situation: he enters the play with a theory worked out in detail. Even if this may not be the case for a particular player, it is clear that such an assumption will certainly not spoil his chances.

Accordingly, we define the "strategy" of a player  $S_m$  as follows:

In order to describe completely the "strategy" of a player  $S_m$  ( $m = 1, 2, \dots, n$ ) the following specifications are necessary:

As before, let  $S_m$  have the "steps" with the numbers  $v_1^{(m)}, v_2^{(m)}, \dots, v_{\sigma_m}^{(m)}$  and assume that at the moment when he decides on the  $v_k^{(m)}$ -th "step" ( $k = 1, 2, \dots, \sigma_m$ ) the results of the "draws" and "steps" with the numbers  $\bar{\mu}_1^{(m,k)}, \bar{\mu}_2^{(m,k)}, \dots, \bar{\mu}_{\alpha_{m,k}}^{(m,k)}$  and  $\bar{v}_1^{(m,k)}, \bar{v}_2^{(m,k)}, \dots, \bar{v}_{\beta_{m,k}}^{(m,k)}$  respectively are available to him, that is, they are "earlier" than  $v_k^{(m)}$ . For each possible combination of results of the "draws" and "steps" mentioned above (obviously, there is only a finite number of such combinations) it must be specified what  $S_m$ 's decision with respect to the  $v_k^{(m)}$ -th "step" is going to be (i.e., what will be the outcome of this step).

One sees immediately that only a finite number of strategies is available to  $S_m$ , which we shall call  $S_1^{(m)}, S_2^{(m)}, \dots, S_{\Sigma_m}^{(m)}$ .

It can now easily be shown (using the assumption on the absence of cycles in the Introduction, i., definition of a game of strategy, (8)) that the course of a play is described in a permissible and unambiguous manner if we specify

FIGURE 48. A page of [79] defining "strategy"

## 17. SOME QUESTIONS

**17.1.** The beauty about games is that they provide an endless source for problems. In general, for any family of games like like  $n \times m \times k$  Rubik cubes or  $n \times m$  tic-tac-toe games, one can ask asymptotic questions or extend things to in higher dimensions. Four dimensional Rubik cubes based on tesseract have been implemented on computers in the 90's already. Given a puzzle( $n$ ), not only the growth rate of  $God(puzzle(n))$  but the construction of the solution path is interesting.

**17.2.** Problem A) Is there a closed formula for the god number for the transposition game of the wheel graphs  $W_n$ ? Are the god numbers of  $W_n$  and  $S_n$  the same for large  $n$  or are there infinitely large  $n$ , where  $God(S_n) = \lfloor 3n/2 \rfloor > God(W_n)$ ? For the star graph, see [1] (where a proof sketch appears in Theorem 7.) Unlike for path graphs  $P_n$  or circular graphs  $C_n$ , where the god number grows quadratically with  $n$ , the god number grows linearly for wheel or star graphs because the diameter of such graphs is 2. If a graph has diameter  $O(n)$  like for  $P_n$  or  $S_n$ , the God number is expected to grow like  $O(n^2)$ .

**17.3.** Problem B) Given a random connected graph in Erdos-Renyi probability space, or a random tree obtained by some branching process. How does the God number of the transposition game graph or the sliding game graph behave in average? In particular, how does the expected God number behave in the limit  $n \rightarrow \infty$ ? The answer will depend on the parameters of the probability models.

**17.4.** Problem C) Sliding or transposition problems look interesting on cylinder  $C_n \times P_m$  or torus  $C_n \times C_n$ . One can play such puzzles on any discrete manifold. We have seen that the god number different on a Möbius strip or a cylinder even so the Cayley graphs  $G_1, G_2$  have the same vertex degree and same size. It would be interesting to study the topological properties of these graphs. They have the same size but different diameter.

**17.5.** Problem D) It would be nice to have more examples of non-square  $n \times m$  chess boards set-ups where the game graph has a reasonable size. There are infinitely many variants one can consider like on 3D cubes or on different manifolds such as tori. If we take a standard chess and move each army of 16 pieces one step forward, it can be played on a torus and is mate in 1. There are likely also other games situations with unexpected outcome. Take a standard chess for example and play it on the Moebius strip where left and right part are identified in an opposite way. White can as a first move immediately capture a rook, but then also black can do that. What happens on a  $8 \times 4$  board with the standard set-up with the middle 4 rows taken away so that all figures are covered. Now play this on the Moebius strip.

**17.6.** Problem E) Mankala, where each player has exactly  $m$  stones in each of the  $n = 6$  pits has a chance to be analyzed for small  $n$  and  $m$ . For  $n = 1$  and 2, the first player obviously wins. For a larger number of pits, unexpected strategies might occur due to the rule that if one has a pit with a lot of stones so that one can go around the entire game and land in an empty of its own pits, one gets all the stones from the opponent. For small  $n$ , one can draw out the game graphs.

**17.7.** Problem F) What happens with the 4 peg tower of Hanoi (Reeves puzzle) in the limit  $n \rightarrow \infty$ . Does the shape also converge to a fractal object if realized suitably in  $\mathbb{R}^3$ , similarly as the Hanoi graph converges to a Sierpinski fractal in  $\mathbb{R}^2$ . In the Hanoi case, the geodesic is the boundary line when embedded as the Sierpinski fractal. What are the geodesics in the Reeves puzzle?

**17.8.** Problem G) For transposition puzzles, there is the interesting question about the size of the **antipodal set**, the set of points which have maximal distance to some vertex. In any Cayley graph, this is independent of the vertex. In general the **BFS layer profile** is of interest. This is the length of the wave front starting at  $v$ . This layer profile is most of the time concave down (unimodal) but we have seen exceptions. What BFS layer profiles can occur? A concrete question to ask is whether the BFS layer profile determines the graph  $\Gamma$  in a transposition graph set-up or whether there are non-isomorphic graphs  $\Gamma$  with  $n$  vertices such that their BFS layer profiles are the same.

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