LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 35: General Stokes

Introduction

35.1. Having seen a fundamental theorem (FTC) in dimension 1, two theorems (FTLI, GREEN) in dimension two and three theorems (FTLI,STOKES,GAUSS) in dimension 3, we expect 4 theorems in dimensions 4. This is indeed the case, but how do we formulate such a theory? How would you formulate this in 4 dimensions where points have coordinates (x, y, z, w)?

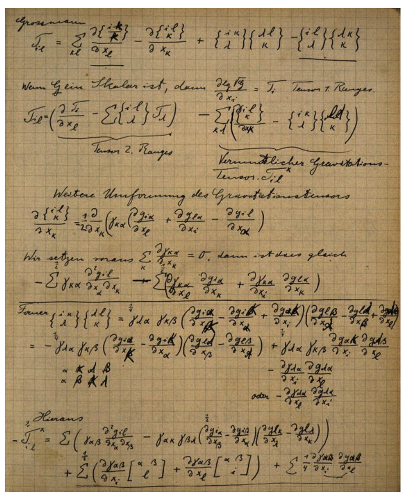

FIGURE 1. A page from Einstein's Zürich notebook features tensors.

35.2. Élie Cartan introduced **forms**. In three dimensions, a **0-form** is just a scalar function $f(x, y, z)$. A **1-form** is $F = Pdx + Qdy + Rdz$, where P, Q, R are scalar functions and dx, dy, dz are formal expressions. A **2-form** is an expression of the form $F = Pdydz + Qdzdx + Rdxdy$, where dx, dy, dz are again symbols but satisfy rules like $dxdy = -dydx, dxdz = -dzdx$ and $dydz = -dzdy$. A **3-form** finally is written as $fdxdydz$, where $dxdydz$ as a volume form. Most calculus books treat 0-forms and 3-form f as a scalar functions and 1-forms and 2-forms as vector fields. But what is dx? It is a linear map from $\mathbb{R}^3 \to \mathbb{R}$ which maps a vector $[v_1, v_2, v_3]$ to v_1. The expression $dxdy$ as a multi-linear anti-symmetric map from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$: the object $dxdy$ assigns to two vectors v, w the determinant of the matrix $v, w, [0, 0, 1]^T$ as column vectors. which is equal to $v \times w \cdot k = v_1 w_2 - v_2 w_1$. Switching v and w changes the sign so that $dxdy = -dydx$ and especially $dxdx = 0$. The object $dxdydz$ is a multi-linear map from $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ assigning to 3 vectors u, v, w the determinant of the matrix in which u, v, w are the columns. Again, switching two elements changes the sign. For example $dxdydz = -dxdzdy$, or $dxdxdz = 0$.

35.3. Breaking away from notions like cross product, we get now objects which can be defined in arbitrary dimensions $\mathbb{R}^n$. A **k-form** is a rule which at every point defines a multi-linear and anti-symmetric map to the reals. Let us look how this is defined in 4 dimensions: a **0**-form is a scalar function f. It assigns to every point (x, y, z, w) a number $f(x, y, z, w)$. A **1-form** is an expression $F = Pdx + Qdy + Rdz + Sdw$ which can be thought of as a vector field $F = [P, Q, R, S]$. A **2-form** is an expression $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydw + Rdzdw$. It is a field with 6 components. A **3-form** is an expression $F = Adydzdw + Bdxdzdw + Cdxdydw + Ddxdydz$. As it is a field with 4 components we can again see it as a "vector field". A **4-form** is an expression $F = fdxdydzdw$. As it has only one component, we can again think of it as a "scalar function" even so this a **lie**. A 4-form is a different object than a 0-form.

35.4. The **exterior derivative** produces from a k form a $(k+1)$-form. First define the 1-form $df = f_x dx + f_y dy + f_z dz + f_w dw$ for a 0-form f, then use this for general k forms. Given a 1-form $F = Pdx + Qdy + Rdz + Sdw$ define $dF = (P_x dx + P_y dy + P_z dz + P_w dw)dx + (Q_x dx + Q_y dy + Q_z dz + Q_w dw)dy + (R_x dx + R_y dy + R_z dz + R_w dw)dz + (S_x dx + S_y dy + S_z dz + S_w dw)dw$ which simplifies to $dF = (Q_x - P_y)dxdy + (R_x - P_z)dxdz + (S_x - P_w)dxdw + (R_y - Q_z)dydz + (S_y - Q_w)dydw + (R_w - S_z)dwdz$. If $F = Adxdy + Bdxdz + Cdxdw + Pdydz + Qdydw + Rdzdw$ is 2-form, then $dF = (A_x dx + A_y dy + A_z dz + A_w dw)dxdy + (B_x dx + B_y dy + B_z dz + B_w dw)dxdz + (C_x dx + C_y dy + C_z dz + C_w dw)dxdw + (P_x dx + P_y dy + P_z dz + P_w dw)dydz + (Q_x dx + Q_y dy + Q_z dz + Q_w dw)dydw + (R_x dx + R_y dy + R_z dz + R_w dw)dzdw$ simplifies to $(P_x - B_y + A_z)dxdydz + (Q_x - C_y + A_w)dxdydw + (R_x - C_z + B_w)dxdzdw + (R_y - Q_z + P_w)dydzdw$. Finally for $F = Adydzdw + Bdxdzdw + Cdxdydw + Ddxdydz$ we have $dF = (A_x dx + A_y dy + A_z dz + A_w dw)dydzdw + (B_x dx + B_y dy + B_z dz + B_w dw)dxdzdw + (C_x dx + C_y dy + C_z dz + C_w dw)dxdydw + (D_x dx + D_y dy + D_z dz + D_w dw)dxdydz = (A_x + B_y + C_z + D_w)dxdydzdw$.

35.5. We can integrate a $(k+1)$-form dF over a $(k+1)$-manifold G and a k-form F over the k-manifold dG, the boundary dG of G. We write $\int_G dF$. To see the general Stokes theorem $\boxed{\int_G dF = \int_{dG} F}$, we need to know that a tensor is. **Machine learning** can justify to introduce the concept. [1] Let E be a space of column vectors and E^* a space of row vectors.

A (p, q)-tensor on E as a multi-linear map from $(E^*)^p \times (E^q)$ to $\mathbb{R}$.

Column vectors are tensors of the type $(1, 0)$, row vectors are tensors of the type $(0, 1)$, matrices are tensors of the type $(1, 1)$. The k-th Jacobean derivative of a function f is a tensor of type $(0, k)$. A tensor of type $(0, 3)$ for example as a 3-dimensional array of numbers A_{ijk}. It defines a multi-linear map assigning to every triplet of vectors u, v, w the number $\sum_{i,j,k} A_{ijk} u^i v^j v^k$. [2] A k-form on a manifold attaches a $(0, k)$ tensor at every point.

[1] There is a "tensor flow" library for example.

[2] **Albert Einstein** would just write $A_{ijk} u^i v^j v^k$ and not bother about the summation symbol.

35.6. $E = \mathbb{R}^n = M(n,1)$ is the space of **column vectors**. Its **dual** $E^* = M(1,n)$ is the space of **row vectors**. To get more general objects we treat vectors as **maps**. A row vector is a linear map $F : E \to \mathbb{R}$ defined by $F(u) = Fu$ and a column vector defines a linear map $F : E^* \to \mathbb{R}$ by $F(u) = uF$. A map $F(x_1, \dots, x_n)$ of several variables is called **multi-linear**, if it is linear in each coordinate. The set $T^p_q(E)$ of all **multi-linear maps** $F : (E^*)^p \times E^q \to \mathbb{R}$ is the space of **tensors of type** (p,q). We have $T^1_0(E) = E$ and $T^0_1(E) = E^*$. The space $T^1_1(E)$ can naturally be identified with the space $M(n,n)$ of $n \times n$ matrices. Indeed, given a matrix A, a column vector $v \in E$ and a row vector $w \in E^*$, we get the bi-linear map $F(v,w) = wAv$. It is linear in v and in w. In other words, it is a tensor of type $(1,1)$.

35.7. Let $\Lambda^q(E)$ be the subspace of $T^0_q(E)$ which consists of tensors F of type $(0,q)$ such that $F(x_1, \dots x_q)$ is anti-symmetric in $x_1, \dots, x_q \in E$: this means $F(x_{\sigma(1)}, \dots, x_{\sigma(q)}) = (-1)^\sigma f(x_1, \dots, x_q)$ for all $i,j = 1, \dots, q$, where $(-1)^\sigma$ is the **sign** of the permutation σ of $\{1, \dots, n\}$. If the **Binomial coefficient** $B(n,q) = n!/(q!(n-q)!)$ counts the number of subsets with q elements $i_1 < \cdots < i_q$ of $\{1, \dots, n\}$ and E has dimension n, then $\Lambda^q(E)$ has dimension $B(n,q)$. A map $F : E \to T^p_q(E)$ is called a (p,q)**-tensor field**. The set $T^1_0(E)$ is the space of **vector fields**. If $g : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map, then $F = d^k g$ is a tensor field of type $(0,k)$. A k**-form** is a $(0,k)$-tensor field F with $F(x) \in \Lambda^k(E)$. A 2-form in $\mathbb{R}^3$ for example attaches to $x \in \mathbb{R}^3$ a bi-linear, anti-symmetric map $F(x)(u,v) = -F(x)(v,u)$. One writes $Pdydz + Qdxdz + Rdxdy$ where $dydz(u,v) = u_2v_3 - u_3v_2$, $dxdz(u,v) = u_1v_3 - u_3v_1$, $dxdy(u,v) = u_1v_2 - v_1u_2$.

35.8. The **exterior derivative** $d : \Lambda^p \to \Lambda^{p+1}$ is defined for $f \in \Lambda^0$ as $df = f_{x_1}dx_1 + \cdots + f_{x_n}dx_n$ and $d(fdx_{i_1} \cdots dx_{i_p}) = \sum_i f_{x_i} dx_i dx_{i_1} \cdots dx_{i_p}$. For $F = Pdx + Qdy$ for example, it is $(P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy = (Q_x - P_y)dxdy$ which is the **curl** of F. If $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$ is a parametrization, then $S = r(G)$ is a m**-surface** and $\delta S = r(\delta G)$ is its **boundary** in $\mathbb{R}^n$. If $F \in \Lambda^p(\mathbb{R}^n)$ is a p**-form** on $\mathbb{R}^n$, then $r^*F(x)(u_1, \dots, u_p) = F(r(x))(dr(x)(u_1), dr(x)(u_2), \dots, dr(x)(u_p))$ is a p-form in $\mathbb{R}^m$ called the **pull-back** of r. Given a p-form F and an p-surface $S = r(G)$, define the integral $\int_S F = \int_G r^*F$. The **general Stokes theorem** is

Theorem: $\int_S dF = \int_{\delta S} F$ for a $(m-1)$-form F and m surface S in E.

35.9. Proof. As in the proof of the divergence theorem, we can assume that the region G is simultaneously of the form $g_j(x_1, \dots, \hat{x}_j, \dots x_m) \le x_j \le h_j(x_1, \dots, \hat{x}_j, \dots x_m)$, where $1 \le j \le n$ and that $F = [0, \dots, 0, F_j, 0, \dots, 0]$. The coordinate independent definition of dF reduces the result to the divergence theorem in G. QED

EXAMPLES

35.10. For $n = 1$, there are only 0-forms and 1-forms. Both are scalar functions. We write f for a 0-form and $F = fdx$ for a 1-form. The symbol dx abbreviates the linear map $dx(u) = u$. The 1-form assigns to every point the linear map $f(x)dx(u) = f(x)u$. The exterior derivative $d : \Lambda^0 \to \Lambda^1$ is given by $df(x)u = f'(x)u$. Stokes theorem is the **fundamental theorem of calculus** $\int_a^b f'(x)dx = f(b) - f(a)$.

35.11. For $n = 2$, there are 0-forms, 1-forms and 2-forms. It is custom to write $F = Pdx + Qdy$ rather than $F = [P, Q]$ which is thought of as a linear map $F(x, y)(u) = P(x, y)u_1 + Q(x, y)u_2$. A 2-form is also written as $F = f dxdy$ or $F = f dx \wedge dy$. Here $dxdy$ means the bi-linear map $dxdy(u, v) = (u_1v_2 - u_2v_1)$. The 2-form defines such a bi-linear map at every point (x, y). The exterior derivative $d\Lambda^0 \to \Lambda^1$ is $df(x, y)(u_1, u_2) = f_x(x, y)u_1 + f_y(x, y)u_2$ which encodes the Jacobian $df = [f_x, f_y]$, a row vector. The exterior derivative of a 1-form $F = Pdx + Qdy$ is $dF(x, y)(u, v) = (-1)^1 P_y(x, y) \det([u, v]) + (-1)^2 Q_x(x, y) \det([u, v])$ which is $(Q_x - P_y)dxdy$. Using coordinates is convenient as $dF = P_ydydx + Q_xdxdy = (Q_x - P_y)dxdy$ using now that $dydx = -dxdy$.

35.12. For $n = 3$, we write $F = Pdx + Qdy + Rdz$ for a 1-form, and $F = Pdydz + Qdzdx + Rdxdy$ for a 2-form. Here $dydz = dy \wedge dz$ are symbols representing bilinear maps like $dydz(u, v) = u_2v_3 - v_3u_2$. As a 2-form has 3 components, it can be visualized as vector field. A 3-form $f dxdydz$ defines a scalar function f. The symbol $dxdydz = dx \wedge dy \wedge dz$ represents the map $dxdydz(u, v, w) = \det([uvw])$. The exterior derivative of a 1-form gives the curl because $d(Pdx + Qdy + Rdz) = P_ydydx + P_zdzdx + Q_xdxdy + Q_zdzdy + R_xdxdz + R_ydydz$ which is $(R_y - Q_z)dydz + (P_z - R_x)dzdx + (Q_x - P_y)dxdy$. The exterior derivative of a 2-form $Pdydz + Qdzdx + Rdxdy$ is $P_xdxdydz + Q_ydydzdx + R_zdzdxdy = (P_x + Q_y + R_z)dxdydz$. To integrate a 2-form $F = x^2yzdxdy + yzdydz + xzdxdz$ over a surface $r(u, v) = [x, y, z] = [uv, u - v, u + v]$ with $G = \{u^2 + v^2 \leq 1\}$ we end up with integrating $F(r(u, v)) \cdot r_u \times r_v$. In order to integrate dF for a 1-form $F = Pdx + Qdy + Rdz$ we can also pull back F and get $\iint_G F_v(r(u, v))r_u - F_u(r(u, v)r_v \, dudv$.

35.13. For $n = 4$, where we have 0-forms f, 1-forms $F = Pdx + Qdy + Rdz + Sdw$ and 2-forms $F = F_{12}dxdy + F_{13}dxdz + F_{14}dxdw + F_{23}dydz + F_{24}dydw + F_{34}dzdw$ which are objects with 6 components. Then 3-forms $F = Pdydzdw + Qdxdzdw + Rdxdydw + Sdxdydz$ and finally 4-forms $f dxdydzdw$.

Remarks

35.14. Historically, differential forms emerged in 1922 with Élie Cartan. Most textbooks introduce the Grassmannian algebra early and use the language of "chains" for example which is the language used in algebraic topology. I myself taught the subject in this old-fashioned way too, back in 1995. [3] It was Jean Dieudonné in 1972 who freed the general Stokes theorem from chains and used first the coordinate free pull back idea. This allowed us in this lecture to formulate the general Stokes theorem from scratch **on a single page** with all definitions.

35.15. What is a differential form? We have seen a mathematically precise definition: a differential form is a **kind of field**: it defines a multi-linear anti-symmetric function that is attached to each point of space. But what is the intuition and what are ways to "visualize" and "see" and "understand" such an object? Here are four paths. Maybe one of them helps:

[3] Caltech notes: https://people.math.harvard.edu/knill/teaching/math109_1995/geometry.pdf

A) Using **Stokes** one can see a form as a functional F, which assigns to a m-dimensional oriented surface S a number $\int_S F \cdot dS$ such that $\int_{-S} F \cdot dS = \int_S (-F) \cdot dS = -\int_S F \cdot dS$. [4] This way of thinking about forms matches what we do in the discrete. If we have a k-form on a graph, then this is a function on k-dimensional oriented complete subgraphs. Given a graph S we have $\int_S F \cdot dS = \sum_{x \in S} F(x)$, where the sum is over all k-dimensional simplices in S.

B) One can understand differential forms better using arithmetic, the **Grassmannian algebra**. This is done with the help of the **tensor product**, which induces an **exterior product** $F \wedge G$ on $\Lambda^p \times \Lambda^q \to \Lambda^{p+q}$. This product generalizes the cross product $\Lambda^1 \times \Lambda^1 \to \Lambda^2$ which works for $n = 3$ as there, the space of 1-forms Λ^1 and 2-forms Λ^2 can be identified. The exterior algebra structure helps to understand k-forms. We can for example see a 2-form as an exterior product $F \wedge G$ of two 1-forms. We can think of a 2-form for example as attaching two vectors at a point and identify two such frames if their orientation and parallelogram areas match.

C) A third way comes through **physics**. We are familiar with manifestations of electomagnetism: we see light, we use magnets to attach papers to the fridge or have magnetic forces keep the laptop lid closed. Electric fields are felt when combing the hair, as we see sparks generated by the high electric field obtained by stripping away the electrons from the head. We use magnetic fields to store information on hard drives and electric fields to store information on a SSD hard drive. Non-visible electro-magnetic fields are used when communicating using cell phones or connecting through blue-tooth or wireless network connections. The electro-magnetic field E, B is actually a 2-form in 4-dimensions. The $B(4,2) = 6$ components are $(E_1, E_2, E_3, B_1, B_2, B_3)$.

D) A fourth way comes through **discretization**. When formulating Stokes on a discrete network, everything is much easier: a k-form is just a function on oriented k-dimensional complete subgraphs of a network. Start with a graph $G = (V, E)$ and orient the complete subgraphs arbitrarily. Given a k-form F, a function on k-simplices has an exterior derivative at a $k+1$ dimensional simplex x is defined as $dF(x) = \sum_{y \subset x} \sigma(y,x) F(y)$, where the sum is over all k-dimensional sub-simplices of x and $\sigma(y,x) = 1$ if the orientation of y matches the orientation of x or -1 else. We have for example seen that for a 1-form F, a function on edges, the exterior derivative at a triangle x is the sum over the F values of the edges, where we add up the value negatively if the arrow of the edge does not match the orientation of the triangle.

Applications

35.16. An **electromagnetic field** is determined by a 1-form A in 4-dimensional space time. The electromagnetic field is $F = dA$. The Maxwell equations are $dF = 0$ (the relation $d \circ d = 0$ is seen in the homework). The second part of the Maxwell equations are $d^*F = j$, where $d^* : \Lambda^p \to \Lambda^{p-1}$ is the adjoint and j is a 1-form encoding both the electric charge and the electric current. We can always gauge with a gradient $A + df$ so that $d^*(A + df) = 0$ (Coulomb gauge). Using $d^*A = 0$, the Maxwell equations reduced to the Poisson equation $LA = (dd^* + d^*d)A = j$, where L is the **Laplacian** on

[4] David Bachman's text on differential forms: "it is a thing which can be integrated".

1-forms. The electric current j defines the electromagnetic field F simply by inverting the Laplacian. This is a bit tricky in the continuum, as the inverse is an integral operator. [5] In the discrete it is just the inverse of the matrix L, which by the way is always an invertible $|E| \times |E|$ matrix if the graph $G = (V, E)$ is simply connected. And there was light!

Homework

Problem 35.1: Given the 1-form $F(x, y, z, w) = [x^3, y^5, z^5, w^2] = x^3 dx + y^5 dy + z^5 dz + w^2 dw$ and the curve $C : r(t) = [\cos(t), \sin(t), \cos(t), \sin(t)]$ with $0 \leq t \leq \pi$. Find the line integral $\int_C F(r(t)) \cdot dr$.

Problem 35.2: Given the 1-form $F = [xyz, xy, wx, wxy] = xyzdx + xydy + wxdz + wxydw$, find the curl dF. Now find $\iint_S dF$ over the 2-dimensional surface $S : x^2 + y^2 \leq 1, z = 1, w = 1$ which has as a boundary the curve $C : r(t) = [\cos(t), \sin(t), 1, 1]^T$, $0 \leq t \leq 2\pi$. You certainly can use the Stokes theorem. If you like to compute both sides of the theorem you can see how the theorem works. The 2-manifold S is parametrized by $r(t, s) = [s, t, 1, 1]^T$. The $(r_s \wedge r_t)_{ij}$ has 6 components, where only one component $(r_s \wedge r_t)_{12}$ is nonzero. This will match with the $dF_{12} = Pdxdy$ part of the 6-component 2-form dF building the curl. We will have to integrate then over $G = s^2 + t^2 \leq 1$.

Problem 35.3: Given the 2-form $F = z^4 xdxdz + xyzw^2 dydw$ and the 3-sphere $x^2 + y^2 + z^2 + w^2 = 1$ oriented outwards. What is the integral $\iiint_S dF$? To compute this 3D integral, you can use the general integral theorem.

Problem 35.4: Given the 3-form $F = xyzdxdydz + y^2 zdydzdw$, find the divergence dF. Now find the flux of F through the unit sphere $x^2 + y^2 + z^2 + w^2 = 1$ oriented outwards.

Problem 35.5:
a) Take $f(x, y, z, w)$. Check that $F = df$ satisfies $dF = 0$.
b) Take $F = F_1 dx + F_2 dy + F_3 dz + F_4 dw$. Compute the curl $G = dF$ and check that $dG = 0$.
c) Take the 2-form $F = F_{12} dxdy + F_{13} dxdz + F_{14} dxdw + F_{23} dydz + F_{24} dydw + F_{34} dzdw$. Write down the 3-form $G = dF$ and check $dG = 0$.
d) Take the 3-form $F = F_1 dydzdw + F_2 dxdzdw + F_3 dxdydw + F_4 dxdydz$ and compute the 4-form $G = dF$. Check that $dG = 0$.

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[5] There are thick books about this like Jackson's Electromagnetism, the bible of the topic.