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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 38: Final Exam Practice

Problems

Problem 38P.1) (10 points):
a) Write down the Laplacian L of the complete graph with four vertices and find its eigenvalues and eigenvectors.
b) Solve the wave equation $f_{tt} = -Lf$ on the graph with initial position $f(0) = [2, 4, 3, 3]$.

Problem 38P.2) (10 points):
a) The set of all 800×600 gray scale pictures are a subspace of a large vector space $M(m, n)$ What is the dimension of this space?
b) The map $(x, y, z, w) \to (y, z, w, x)$ defines a linear transformation on $\mathbb{R}^4$. What is its determinant?
c) The set of all words in the alphabet $A - Z$ feature an addition $+$ given by concatenation. The zero element 0 is the empty word. What is the name of this algebraic structure?
d) Which of the three following people is not associated to an axiom system *Euclid*, *Peano*, *Jordan*.
e) What does the rank-nullity theorem tell for a $n \times m$ matrix?
f) If S is the eigenbasis of a $n \times n$ matrix A, what can you say about $S^{-1}AS$?
g) If A is a $n \times 1$ matrix and $A = QR$ is the QR-decomposition of A, what shape does the matrix R have?
h) What is the name of a matrix A with complex entries for which $\overline{A}^T A = 1$.
i) What is the determinant of a matrix $A \in SU(2)$?
j) Which of the fundamental forces are associated to $SU(2)$. The electromagnetic, the weak, the strong or the gravitational force?

Problem 38P.3) (10 points):
We look for 4 numbers x, y, z, w. We know their sum is 20 and that their "super sum" $x - y + z - w$ is 10. As a matter of fact these two equations form a system $Ax = b$ which defines a 2-dimensional plane V in 4-dimensional space.
a) (6 points) Find the solution space of all these numbers by row reducing its augmented matrix $B = [A|b]$ carefully.

$$B = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 20 \\ 1 & -1 & 1 & -1 & 10 \end{array}\right] .$$

b) (4 points) Find two linearly independent vectors which are perpendicular to the kernel of A.

Problem 38P.4) (10 points):
People on social media have been in war about expressions like $\boxed{2x/3y - 1 \text{ if } x = 9 \text{ and } y = 2}$. Computers and humans disagree: most humans get 2, while most machines return 11. A psychologist investigates whether the size of the numbers influences the answer and asks people. This needs data fitting: using the least square method, find those a and b such that

$$\frac{ax}{3y} - b = 2$$

best fits the data points in the following table:

x	y
9	3
6	1
-3	1
0	1

Problem 38P.5) (10 points):

Let $\vec{x} = \begin{bmatrix} v \\ e \\ f \end{bmatrix}$ denote the number of vertices, edges and faces of a polyhedron. During a **Barycentric refinement**, this vector transforms as

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \vec{x} \, .$$

a) (5 points) Verify that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ are eigenvectors of A and find their eigenvalues.

b) (5 points) Write down a closed form solution of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with the initial condition $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix}$.

Problem 38P.6) (10 points):

The **Arnold cat map** is $T\vec{v} = A\vec{v}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \, .$$

It is an icon of chaos theory.

a) (2 points) What is the characteristic polynomial of A?

b) (2 points) Find the eigenvalues of A.

c) (2 points) Find the eigenvectors of A.

d) (2 points) Is the discrete dynamical system defined by A asymptotically stable or not?

e) (2 points) Write down an orthogonal matrix S and a diagonal matrix B such that $B = S^{-1}AS$.

Problem 38P.7) (10 points):
The following configuration is called the "Beacon Oscillator" in the **Game of Life**.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (2 points) What is the rank and the nullity of A?
b) (4 points) Find a basis for the kernel and a basis for the image of A.
c) (4 points) The following matrix is called the "glider configuration" in the **Game of life**.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Find the inverse of A using row reduction.

Problem 38P.8) (10 points):
Remember to give computation details. Answers alone can not be given credit.
a) (2 points) The following matrix displays the solution of the Cellular automaton 10. Find its determinant

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 1 & 2 & 3 & 8 & 8 \\ 4 & 5 & 0 & 8 & 8 \\ 6 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 11 & 2 & 3 & 2 & 1 \\ 1 & 12 & 3 & 2 & 1 \\ 1 & 2 & 13 & 2 & 1 \\ 1 & 2 & 3 & 12 & 1 \\ 1 & 2 & 3 & 2 & 11 \end{bmatrix}.$$

e) (2 points) Find the determinant of $E = 2Q + 5Q^{-1} + 7I$: (you can leave it in terms of eigenvalues of the basic circulant matrix Q you have seen. No simplifications are required):

$$E = \begin{bmatrix} 7 & 2 & 0 & 0 & 5 \\ 5 & 7 & 2 & 0 & 0 \\ 0 & 5 & 7 & 2 & 0 \\ 0 & 0 & 5 & 7 & 2 \\ 2 & 0 & 0 & 5 & 7 \end{bmatrix}.$$

Problem 38P.9) (10 points):
Find the general solution to the following differential equations:
a) (1 point)
$$f'(t) = 1/(t+1)$$
b) (1 point)
$$f''(t) = e^t + t$$
c) (2 points)
$$f''(t) + f(t) = t + 2$$
d) (2 points)
$$f''(t) - 2f'(t) + f(t) = e^t$$
e) (2 points)
$$f''(t) - f(t) = e^t + \sin(t)$$
f) (2 points)
$$f''(t) - f(t) = e^{-3t}$$

Problem 38P.10) (10 points):

We consider the nonlinear system of differential equations

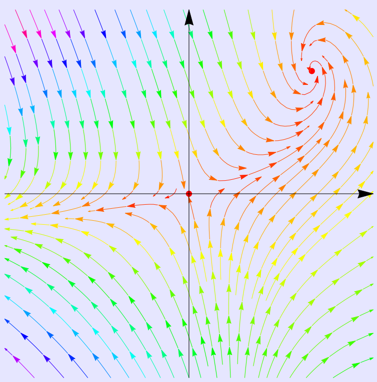
$$\begin{aligned}\frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy\ .\end{aligned}$$

a) (2 points) Find the equilibrium points.

b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobian matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

d) (2 points) Which of the diagrams A-D is the phase portrait of the system above?



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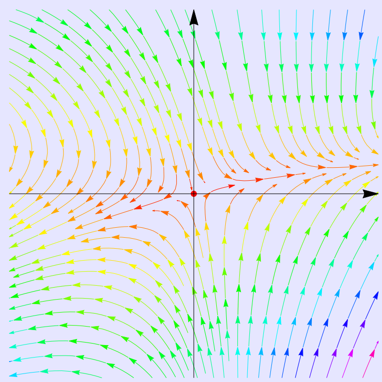

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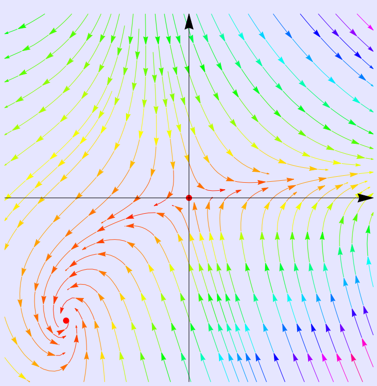

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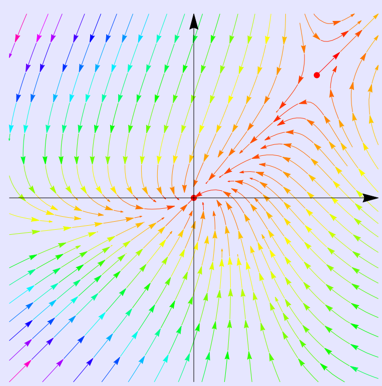

D

Problem 38P.11) (10 points):
a) (6 points) Find the **Fourier series** of the function which is 1 if $|x| > 1$ and -1 else. We call it the **Pacific rim** function.

$$f(x) = \begin{cases} 1 & |x| > 1 \\ -1 & |x| \leq 1 \end{cases} .$$

b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Problem 38P.12) (10 points):
a) Solve the system $f_t = 3f_{xx} - f + t$ with $f(0, x) = x$ on $[-\pi, \pi]$
b) Solve the system $f_t = 3f_{xx} - 9f_{xxxx}$ with $f(0, x) = x$.

Problem 38P.13) (10 points):
a) Solve the system $f_{tt} = 9f_{xx}$ with $f(0, x) = x$ and $f_t(0, x) = 11 \sin(88x)$.
b) Solve $f_{tt} = 9f_{xx} - f_{xxxx}$ with $f_t(0, x) = x$.