Unit 27: Cookbook

Seminar

27.1. There is something nice about **recipes**. They relax, produce **robust** and **reliable** results, boost **confidence** and build a fertile ground for creativity. An important principle at the heart of the learning process is now also appreciated in artificial intelligence:

> Used once, it is a trick. Used twice, it has become method.

27.2. When learning something it is good to follow rules at first. **Understanding** happens in different levels: **knowing** the terms, being able to **use** it, **understanding** its mechanism, be able to **teach** it, and finally to **extend** it. When speaking a new language which can be a programming language too, we first "parrot" to get a feel for the language. It does not make much sense to start reading the grammar book or to read the language specification before one has tried a few examples.

27.3. The same applies when solving mathematical problems or even when building theories. There are rules which help to build theory. There are rules which help to find proofs. There are methods to be creative. In this seminar, we want to illustrate in the context of differential equations, how powerful a "solution manual" can be. This **cookbook** not only allows us to understand the operator method better; it also enables us to solve differential equations in a fraction of the time.

27.4. If you have never been taught to solve a differential equation like $f''(x)+f(x) = x$ you have to be ingenious. You might try to push the **method of separation of variables** further, you might try Taylor expansions and find relations of the Taylor coefficients, you might to find a solution by trial and error. Why not just try $f(x) = x$ for example? Wow, bingo we were lucky. This guess is actually a solution because $f'' = 0$. But this "trial and error" needs in general quite a bit of luck. For $f''(x)+f(x) = x^3$, the function $f(x) = x^3$ no more solves the problem and we have to find another way.

27.5. We have seen in the last lecture a general method to solve differential equations like $f'' - 2f' + f = x^2$. We sometimes use also the variable t instead of x. This is on purpose. If we use t, we indicate that t is **time**. If we use x, we indicate that x is **position**. It does not matter. You need to be aware that we can deal with any variable. Let us review first the operator method.

A) Write the equation in operator form $p(D)f = g$.
B) Factor the polynomial p to get $(D - \lambda_1) \cdots (D - \lambda_n)f = g$.
C) Invert each factor using $(D - \lambda)^{-1}(g)(x) = e^{\lambda x}(\int_0^x e^{-\lambda s} g(s)\, ds + C)$.

27.6. A) In the example, the operator rewrite is $(D^2 - 2D + 1)f = x^2$. b) Factored, it is $(D-1)^2 f = x^2$. B) Invert once to get $(D-1)f = e^x(\int_0^x e^{-s}s^2\, ds + C_1)$ which is (integration by parts) $= C_1e^x + (2e^x - x^2 - 2x - 2)$. Now invert again to get $f = e^x((C_1 + 2)x + C_2 - 6) + x^2 + 4x + 6$. Here is the cookbook method

A) Solve the **homogeneous solution** f_h where the right hand side is 0.
B) Find a **particular solution** f_p using the cookbook.
C) The **general solution** is the sum $f_h + f_p$.

27.7. Example: A) Since the eigenvalue 1 appears twice, we have $f_h(x) = C_1e^x + C_2xe^x$. B) For the particular solution, try $f = Ax^2 + Bx + C$. It gives $f' = 2Ax + B, f'' = 2A$. Plug this into the equation to get $f'' - 2f' + f = 2A - 2(2Ax + B) + Ax^2 + Bx + C = Ax^2 + (B - 4A)x + C + 2A - 2B = x^2$. Comparing coefficients gives $A = 1, B = 4, C = 6$ so that $f_p = x^2 + 4x + 6$. C) The general solution is $x^2 + 4x + 6 + C_1e^x + C_2xe^x$.

Problem A: Do as many examples as possible from the list below. Try first without looking at the solution, then compare.

27.8. The "recipe" for finding solutions is to **feed in the same class of functions which you see on the right** and **multiply with x when stuck**.

$\lambda_1 \neq \lambda_2$ real	$f_h = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$
$\lambda_1 = \lambda_2$ real	$f_h = C_1e^{\lambda_1 x} + C_2xe^{\lambda_1 x}$
$\lambda_1 = ik, \lambda_2 = -ik$	$f_h = C_1\cos(kx) + C_2\sin(kx)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$f_h = C_1e^{ax}\cos(kx) + C_2e^{ax}\sin(kx)$

27.9. Here are examples how to get the particular solution Ansatz:

$g(x) = a$	$f_p(x) = A$ constant
$g(x) = ax + b$	$f_p(x) = Ax + B$
$g(x) = ax^2 + bx + c$	$f_p(x) = Ax^2 + Bx + C$
$g(x) = a\cos(bx)$	$f_p(x) = A\cos(bx) + B\sin(bx)$
$g(x) = a\sin(bx)$	$f_p(x) = A\cos(bx) + B\sin(bx)$

$g(x) = a\cos(bx)$ with $p(D)g = 0$	$f(x) = Ax\cos(bx) + Bx\sin(bx)$
$g(x) = a\sin(bx)$ with $p(D)g = 0$	$f(x) = Ax\cos(bx) + Bx\sin(bx)$
$g(x) = ae^{bx}$	$f(x) = Ae^{bx}$
$g(x) = ae^{bx}$ with $p(D)g = 0$	$f(x) = Axe^{bx}$

EXAMPLE 1: $f'' = \cos(5x)$

This is of the form $D^2 f = g$ and can be solved by inverting D which is integration: integrate a first time to get $Df = C_1 + \sin(5x)/5$. Integrate a second time to get $f = C_2 + C_1 x - \cos(5x)/25$ This is the operator method in the case $\lambda = 0$.

EXAMPLE 2: $f' - 2f = 2x^2 - 1$

This homogeneous differential equation $f' - 2f = 0$ is hardwired to our brain. We know its solution is Ce^{2x}. To get a particular solution, try $f(t) = Ax^2 + Bx + C$. We have to compare coefficients of $f' - 2f = -2Ax^2 + (2A - 2B)x + B - 2C = 2x^2 - 1$. We see that $A = -1, B = -1, C = 0$. The special solution is $-x^2 - x$. The complete solution is $f = -x^2 - x + Ce^{2x}$

EXAMPLE 3: $f' - 2f = e^{2x}$

In this case, the right hand side is in the kernel of the operator $T = D - 2$ in equation $T(f) = g$. The homogeneous solution is the same as in example 2, to find the inhomogeneous solution, try $f(x) = Axe^{2x}$. We get $f' - 2f = Ae^{2x}$ so that $A = 1$. The complete solution is $f = xe^{2x} + Ce^{2x}$

EXAMPLE 4: $f'' - 4f = e^t$

To find the solution of the homogeneous equation $(D^2 - 4)f = 0$, we factor $(D-2)(D+2)f = 0$ and add solutions of $(D-2)f = 0$ and $(D+2)f = 0$ which gives $C_1e^{2t} + C_2e^{-2t}$. To get a special solution, we try Ae^t and get from $f'' - 4f = e^t$ that $A = -1/3$. The complete solution is $f = -e^t/3 + C_1e^{2t} + C_2e^{-2t}$

EXAMPLE 5: $f'' - 4f = e^{2t}$

The homogeneous solution $C_1e^{2t} + C_2e^{-2t}$ is the same as before. To get a special solution, we can not use Ae^{2t} because it is in the kernel of $D^2 - 4$. We try Ate^{2t}, compare coefficients and get $f = te^{2t}/4 + C_1e^{2t} + C_2e^{-2t}$

EXAMPLE 6: $f'' + 4f = e^t$

The homogeneous equation is a harmonic oscillator with solution $C_1\cos(2t) + C_2\sin(2t)$. To get a special solution, we try Ae^t compare coefficients and get $f = e^t/5 + C_1\cos(2t) + C_2\sin(2t)$

EXAMPLE 7: $f'' + 4f = \sin(t)$

The homogeneous solution $C_1\cos(2t) + C_2\sin(2t)$ is the same as in the last example. To get a special solution, we try $A\sin(t) + B\cos(t)$ compare coefficients to get $f = \sin(t)/3 + C_1\cos(2t) + C_2\sin(2t)$

EXAMPLE 8: $f'' + 4f = \sin(2t)$

The solution $C_1\cos(2t) + C_2\sin(2t)$ is as before. To get a special solution, we can not try $A\sin(t)$ because it is in the kernel of the operator. We try $At\sin(2t) + Bt\cos(2t)$ instead and compare coefficients $f = t\sin(2t)/16 - t\cos(2t)/4 + C_1\cos(2t) + C_2\sin(2t)$

EXAMPLE 9: $f'' + 8f' + 16f = \sin(5t)$

The homogeneous solution is $C_1e^{-4t} + C_2te^{-4t}$. To get a special solution, we try $A\sin(5t) + B\cos(5t)$ compare coefficients and get $f = -40\cos(5t)/41^2 + -9\sin(t)/41^2 + C_1e^{-4t} + C_2te^{-4t}$

EXAMPLE 10: $f'' + 8f' + 16f = e^{-4t}$

The homogeneous solution is still $C_1e^{-4t} + C_2te^{-4t}$. To get a special solution, we can not try e^{-4t} nor te^{-4t} because both are in the kernel. Add another t and try with At^2e^{-4t}. $f = t^2e^{-4t}/2 + C_1e^{-4t} + C_2te^{-4t}$

EXAMPLE 11: $f'' + f' + f = e^{-4t}$

By factoring $D^2 + D + 1 = (D - (1 + \sqrt{3}i)/2)(D - (1 - \sqrt{3}i)/2)$ we get the homogeneous solution $C_1e^{-t/2}\cos(\sqrt{3}t/2) + C_2e^{-t/2}\sin(\sqrt{3}t/2)$. For a special solution, try Ae^{-4t}. Comparing coefficients gives $A = 1/13$. $f = e^{-4t}/13 + C_1e^{-t/2}\cos(\sqrt{3}t/2) + C_2e^{-t/2}\sin(\sqrt{3}t/2)$

Homework

Problem 27.1 Find the general solution of $f'' + 225f = e^{44t} + t$.

Problem 27.2 Find the general solution of $f'' + 225f = \cos(15t)$.

Problem 27.3 Find the general solution of $f'' + 225f = \cos(10t)$.

Problem 27.4 Find the general solution to $f'' - 10f' + 25f = 4e^{5t}$.

Problem 27.5 Find the general solution to $f''' - 2f'' - f' + 2f = e^t + e^{-t}$.