Unit 26: Operators

Lecture

26.1. A linear map on a function space X is called an **operator**. An example is $Df(x) = f'(x)$ on $X = C^\infty$. More generally, if $p(x)$ is a polynomial, we can look at $p(D)$ on X and look at differential equations $p(D)f = g$ for an unknown function. This is similar as we solved equations $Ax = b$. Now, the operator $p(D)$ plays the role of the matrix A, and f, g replace vectors x and b.

26.2. With the polynomial $p(x) = x^3 + x$ and $g(x) = x^2 + \sin(x)$ we get the problem $p(D)f = g$ which is $f'''(x) + f' = x^2 + \sin(x)$. The equation $p(D)f = g$ is the analogue of an equation $Ax = b$. For example, if $Df = g$, then f is the **anti-derivative** of g. There is a one-dimensional solution space.

Theorem: $p(D)f = g$ has a $\deg(p)$ dimensional solution space.

26.3. If $g = 0$, we get $p(D)f = 0$ and deal with the kernel of a linear transformation and so a linear space. In general, as in the case $Ax = b$, the solution space is **affine**, it is a translated linear space. For example, the harmonic oscillator $(D^2 + 9)f = 0$ has the solution $C_1 \cos(3t) + C_2 \sin(3t)$. To prove the theorem, we need a lemma:

Lemma: $f(t) = e^{\lambda t}(C + \int_0^t e^{-\lambda s} g(s)\, ds)$ solves $f' - \lambda f = g$.

Proof. We just check that it solves the equation. □

26.4. In other words, we have an explicit formula for the inverse $(D - \lambda)^{-1}$. In the case $\lambda = 0$, the formula is

$$D^{-1}g = \int_0^t g(s)\, ds + C \; .$$

So, we can see the formula as a generalized integration formula. To the proof:

Proof. The fundamental theorem of algebra implies that $p(x) = c(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$. In order to solve $p(D)f = g$, we have

$$f = p(D)^{-1}g = c^{-1}(D - \lambda_1)^{-1} \cdots (D - \lambda_n)^{-1} g \; .$$

Now we can invert one of the operators after the other and each integration introduces a new constant. □

26.5. We have compared $Tf = g$ with solving the system of linear equations $Ax = b$. There are some things which go over, there are other things which don't. For example, we don't have a row reduction process to solve the equation $Tf = g$ in infinite dimensions. We also do in general not have a notion of a determinant of D, at least not in an elementary way.

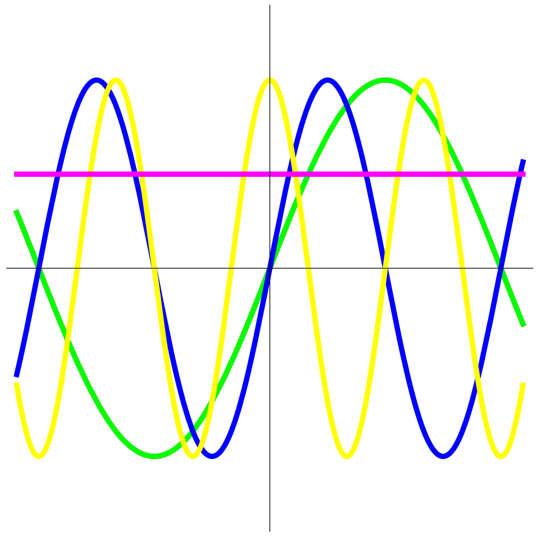

Figure 1. Trigonometric polynomials $\cos(nx), \sin(nx)$ and the constant function 1 all are functions in $C^\infty(\mathbb{T})$ that solve the eigenvalue problem $D^2 f = \lambda f$ for some λ. In the homework you show that these are the only ones and that the eigenvalues of D^2 therefore are in a discrete quantized set.

26.6. The method of solving a differential equation by inverting the operator T is called the **operator method**. It is quite powerful. The advantage is that one does not have to think much: just factor T into linear parts, then apply the inversion formulas. There is a bit more convenient way to do that without computers. We will learn an **engineer method** in Unit 27.

Examples

26.7. To check that an operator is linear, we have to check three things: $T(f+g) = T(f)+T(g)$, $T(\lambda f) = \lambda T(f)$ and $T(0) = 0$ as we did for linear transformations earlier.
a) $T(f)(x) = x^2 f(x-4)$ is linear.
b) $T(f)(x) = f'(x)^2$ is not linear.
c) $T = D^2 + D + 1$ is linear.
d) $T(f)(x) = e^x \int_0^x e^{-t} f(t)\, dt$ is linear.
e) $T(f)(x) = f(\sin(x))$ is linear
f) $T(f)(x) = x^5 f(x^3)$ is linear
g) $T(f)(x) = f(f(x))$ is not linear.
Here is a formal verification for a):
i) $T(0)(x) = x^2 0(x-4) = 0$. ii) $T(f)(x) + T(g)(x) = x^2 f(x-4) + x^2 g(x-4) = x^2[f(x-4)+g(x-4)] = x^2(f+g)(x-4) = T(f+g)(x)$. iii) $T(\lambda f)(x) = x^2 \lambda f(x-4) =$

$\lambda x^2 f(x-4) = \lambda T(f)(x)$.
For a refutation, we need a counter example. For g) for example, take $f(x) = x^3$. Then $T(f)(x) = (x^3)^3 = x^9$. Now, $T(2x^3) = 2(2x^3)^3 = 16x^9$ which is different from $2T(x^3)$.

26.8. The equation $D^3 f = f''' = t^5$ is solved by integrating g three times. This gives $f(x) = x^8/(6*7*8) + C_1 t^2 + C_2 t + C_3$.

26.9. What is the kernel and image of the linear operators $T = D+3$ and $D-2$? Use this to find the kernel of $p(D)$ for $p(x) = x^2 + x - 6$?
The kernel of $T = D+3$ is $\{Ce^{-3x}\}$. The kernel of $D-2$ is $\{Ce^{2x}\}$. The kernel of $(D+3)(D-2)$ contains these two kernels and is $\{Ae^{-3x} + Be^{2x}\}$.

26.10. Verify whether the function $f(x) = e^{-x^2/2}$ is in the kernel of the differential operator $T = D + x$: Solution: We compute $Dg = -xg(x)$. So, $f' + xf(x) = 0$.

26.11. The differential equation $f' - 3f = \sin(x)$ can be written as $Tf = g$ with $T = D-3$ and $g = \sin$. We need to invert the operator T. But we have an inversion formula. We get $f = Ce^{3x} + e^{3x}\int_0^x e^{-3t}\sin(t)\, dt$ which is $Ce^{3x} + (3/10)\sin(x) - \cos(x)/10$. In the proof seminar, we will learn another method to solve this.

Applications

26.12. In quantum mechanics, the operator $P = iD$ is called the **momentum operator** and the operator $Qf(x) = xf(x)$ is called the **position operator**.
Every λ is an eigenvalue of P. The eigenvalue is $e^{i\lambda x}$. What operator is

$$[Q, P] = QP - PQ\ ?$$

You check this in the homework.

26.13. The operator

$$Tf(x) = -f''(x) + x^2 f(x)$$

is called the **energy operator** of the **quantum harmonic oscillator**.
Task: Check that $f(x) = e^{-x^2/2}$ is an eigenfunction of T. What is the eigenvalue?
Solution. Differentiate $f(x) = e^{-x^2/2}$ twice with respect to x. This gives $x^2 f(x) - f(x)$. Since $Tf = f$, the eigenvalue is 1.

26.14. In statistics, one looks at real valued functions f on a probability space Ω. There is a natural linear map from functions to $\mathbb{R}$, which is the **expectation** $\mathrm{E}[f] = \int_\Omega f(x)\, dP(x)$, where P is a probability measure. One usually assumes that these functions are integrable, meaning that $E[|f|] = \int_\Omega |f(x)|\, dP(x) < \infty$ and also that f has the property that $\mathrm{E}[f^2] < \infty$ as this assures that one has finite variance $\mathrm{Var}[f] = \mathrm{E}[(f-m)^2]$. Which of the maps "expectation" or "variance" is linear? Answer: the expectation is a linear map, the variance is not. While $\mathrm{Var}[0] = 0$, it is not true in general that $\mathrm{Var}[f+g] = \mathrm{Var}[f] + \mathrm{Var}[g]$ (an example is $\Omega = [-1, 1]$, $f(x) = x, g(x) = x^3$ then $\mathrm{Var}[f] = \int_{-1}^1 x^2\ dx = 2/3$ and $\mathrm{Var}[g] = \int_{-1}^1 x^6\ dx = 2/7$ and $\mathrm{Var}[f+g] = \int_{-1}^1 (x+x^3)^2\ dx = 184/105$. Even easier to check is that $\mathrm{Var}[2f] = 4\mathrm{Var}[f]$ which verifies that the variance does not scale linearly.

Homework

This homework is due on Tuesday, 4/09/2019.

Problem 26.1: Which of the following are linear transformations?
a) $T(f)(x) = f'''(x)$ on $X = C^\infty(\mathbb{R})$.
b) $T(f)(x) = f'''(x) + 1$ on $X = C^\infty(\mathbb{R})$.
c) $T(f)(x) = x^3 f(x)$ on $X = C^\infty(\mathbb{R})$.
d) $T(f)(x) = f(x)^2$ on $X = C^\infty(\mathbb{R})$.
e) $T(f)(x) = f(0) + \int_0^x f(x)\, dx$ on $X = C(\mathbb{R})$.
f) $T(f)(x) = f(x+2)$ on $X = C^\infty(\mathbb{R})$.
g) $T(f)(x) = \int_{-1}^1 f(x-s)s^2\, ds$ on $C(\mathbb{R})$.
h) $T(f)(x) = f'(x^3) + f(3)$ on $C^\infty(\mathbb{R})$.
i) $T(f)(x) = f'(x^2) - f(5) + 1$ on $C^\infty(\mathbb{R})$.
j) $T(f)(x) = f^2(x)$ on $C(\mathbb{R})$.

Problem 26.2: a) Solve the differential equation $f'' - 5f' + 6f = x^2$ by applying the inversion formula.
b) Solve the differential equation $f'' + f = 0$ using the inversion formula (it will be complex).

Problem 26.3: a) Solve the differential equation $f' - f = e^x$ using the inversion formula.
b) Now find the solution with $f(0) = 4$.

Problem 26.4: The operator $Tf = D^2$ will play an important role throughout the rest of the course. It is the negative of the energy operator P^2. Since $Te^{\lambda x} = \lambda^2 e^{\lambda x}$ we see that every real λ is an eigenvalue of T on $C^\infty(\mathbb{R})$. This completely changes when we look at the same operator on the space $C^\infty(\mathbb{T})$ of 2π periodic functions. Instead of having a continuum of eigenvalues, we will get a discrete set of values. Find all the eigenvalues and eigenvectors of D^2!

Problem 26.5: What is the commutator of the momentum and position operator in quantum mechanics? Remember that $Pf(x) = if'(x)$ and $Qf(x) = xf(x)$. What is $[Q,P] = QP - PQ$? Your result will give the **Heisenberg uncertainty relations**. It has important consequences like that we can not measure both position as well as momentum at the same time.