Unit 23: Nonlinear systems

Lecture

23.1. We look at **nonlinear differential equations** for differentiable functions f, g:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} .$$

In vector form, with $r(t) = [x(t), y(t)]^T$ and vector field $F(x,y) = [f(x,y), g(x,y)]^T$ the differential equation can be written as $r'(t) = F(r(t))$. To say it in the language of vector calculus, we aim to find the **flow lines** of the vector field F. Even having left the linear context, we can still use **linear algebra** to analyze such systems.

23.2. A nonlinear system in population dynamics is the **Murray system**

$$\begin{aligned} x' &= x(6-2x) - xy \\ y' &= y(4-y) - xy . \end{aligned}$$

It is a coupled pair of **logistic systems** which without the xy interaction term would evolve independently of each other. With the interaction, which implements a **competition** situation, we cannot write down a closed-form solutions. Even a computer algebra system is unable to do that for that above Murray system.

23.3. A point (x, y) is called an **equilibrium point** if $F(x, y) = 0$. It is helpful to look for x-**nullclines**, points where $f(x, y) = 0$ and also for y-nullclines, where $g(x, y) = 0$. On x-nullclines, the vector field is vertical, while on y-nullclines, the vector field is horizontal. We are already familiar with the **Jacobian matrix** $A = dF(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$ which is the **linearization**. Given an equilibrium point (x_0, y_0), we get the linearized system $r'(t) = Ar(t)$. We call the equilibrium point **asymptotically stable** if its linearization is asymptotically stable. Here is a simple principle which helps to analyze the flow phase space.

Theorem: If two solution curves cross, we have an equilibrium point.

Proof. This follows from the fact that the system has a unique solution. If two solutions intersect, then there are two solution curves through this point. □

23.4. To analyze a non-linear system, we find the nullclines, the equilibrium points, linearize the system near each equilibrium point, then draw the phase portraits near the equilibrium points and finally connect the dots to see the global phase portrait. Let us do that in the case of the Murray system. Since $f(x,y) = x(6-2x-y)$, the x-nullclines consist of two lines $x = 0$ and $y = 6-2x$. Since $g(x,y) = y(4-x-y)$ the y-nullclines are $y = 0$ and $y = 4-x$. The equilibrium points are $(0,0), (3,0), (0,4), (2,2)$.

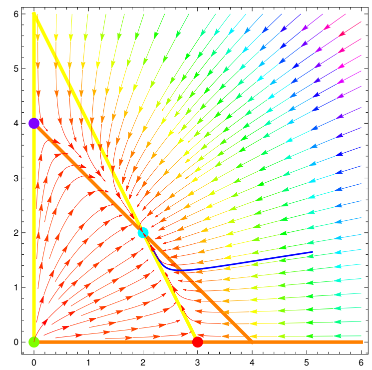

FIGURE 1. The Murray system is a coupled logistic system which describes a **competition model**.

Equilibrium	Jacobian	Eigenvalues	Type
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6,\ \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6,\ \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2,\ \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink

EXAMPLES

23.5. In the 1920's, the **Volterra-Lodka systems** appeared:

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

has equilibrium points $(0,0)$ and $(1/2,1)$. It describes a predator-prey situation like for example a shrimp-shark population. The shrimp population $x(t)$ becomes smaller with more sharks. The shark population grows with more shrimp. Volterra explained so first the oscillation of fish populations in the mediterranean sea

23.6. Given a function $H(x,y)$ of two variables it defines a system

$$\begin{aligned} \dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y) \end{aligned}$$

called Hamiltonian systems. An example is the **pendulum** $H(x,y) = y^2/2 - \cos(x)$ appearing in the homework, in which x denotes the angle between the pendulum and y-axes, and y is the angular velocity, the value $\sin(x)$ is the potential energy. Hamiltonian systems preserve energy $H(x,y)$ because $\frac{d}{dt}H(x(t),y(t)) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$. Orbits stay on level curves of H.

23.7. Lienhard systems are differential equations of the form $\ddot{x}+\dot{x}F'(x)+G'(x) = 0$. With $y = \dot{x} + F(x), G'(x) = g(x)$, this gives

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x) \end{aligned}$$

An example is the **Van der Pol equation** $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical engineering, biology or biochemistry. Since $F(x) = x^3/3 - x, g(x) = x$.

$$\begin{aligned} \dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x \ . \end{aligned}$$

Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a$, $F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \to \infty$ for $x \to \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

23.8. Chaos can occur for systems $\dot{x} = f(x)$ in three dimensions. Here are three examples. They lead already to **strange attractors**.

23.9. The **Roessler system**

$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + y/5 \\ \dot{z} &= 1/5 + xz - 5.7z \end{aligned}$$

23.10. The **Lorentz system**

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3} \end{aligned}$$

23.11. The **Duffing system** models moving plate: $\ddot{x} + \frac{\dot{x}}{10} - x + x^3 - 12\cos(t) = 0$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12\cos(z) \\ \dot{z} &= 1 \end{aligned}$$

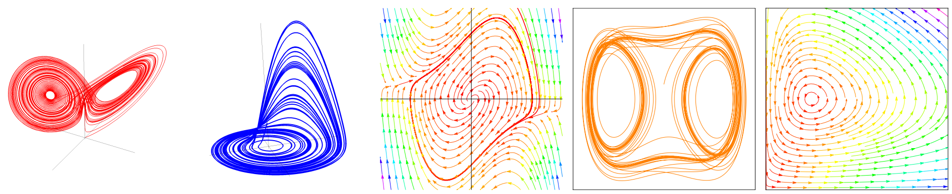

Figure 2. Lorentz, Roessler, Vanderpool, Duffing, Volterra

Homework

Due on Tuesday, 4/02/2019. **Analysis** means 1) find nullclines and equilibria 2) determine stability 3) draw phase space 4) list typical trajectories. Some problems adapted from unpublished notes by Otto Bretscher.

Problem 23.1: Analyze the **Volterra-Lodka system** on $x \geq 0, y \geq 0$:

$$\begin{aligned} \frac{dx}{dt} &= 2x + xy - x^2 \\ \frac{dy}{dt} &= 4y - xy - y^2 \end{aligned}$$

Problem 23.2: Analyze the population model on $x \geq 0, y \geq 0$.

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + 2y - 2) \\ \frac{dy}{dt} &= y(1 - y + 2x - 2) \end{aligned}$$

Problem 23.3: Analyze the **pendulum Hamiltonian system**

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2\sin(x) , \end{aligned}$$

Problem 23.4: Analyze the pendulum with friction

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2\sin(x) - y . \end{aligned}$$

Problem 23.5: Analyze the system

$$\begin{aligned} \frac{dx}{dt} &= x^2 + y^2 - 1 \\ \frac{dy}{dt} &= xy \end{aligned}$$

Oliver Knill, knill@math.harvard.edu, Math 22b, Harvard College, Spring 2019