LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 16: Diagonalization

Lecture

16.1. We say that $\mathcal{B} = \{v_1, v_2, \cdots, v_n\}$ is an **eigenbasis** of a $n \times n$ matrix A if it is a basis of $\mathbb{R}^n$ and every vector $v_1, \dots, v_n$ is an eigenvector of A. The matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$ for example has the eigenbasis $\mathcal{B} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix}\}$. The basis might not be unique. The identity matrix for example has **every basis** of $\mathbb{R}^n$ as eigenbasis.

16.2. Does every matrix have an eigenbasis? One could conjecture it and try to prove it, but one would fail. The answer is no! How do we find a counter example? Remember the **magic matrix** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? Its characteristic polynomial is $p_A(\lambda) = \lambda^2$ so that $\lambda_1 = 0, \lambda_2 = 0$ are the eigenvalues of A. The eigenvectors are in the kernel of A which is one-dimensional only as A has only one free variable. For a basis, we would need two linearly independent eigenvectors to the eigenvalue 0.

16.3. We say a matrix A is **diagonalizable** if it is similar to a diagonal matrix. This means that there exists an invertible matrix S such that $B = S^{-1}AS$ is diagonal. Remember that we often have created transformations like a reflection or projection at a subspace by choosing a suitable basis and diagonal matrix B, then get the similar matrix A.

Theorem: A is diagonalizable if and only if A has an eigenbasis.

Proof. Assume first that A has an eigenbasis $\{v_1, \cdots v_n\}$. Let S be the matrix which contains these vectors as column vectors. Define $B = S^{-1}AS$. Since

$$Be_k = S^{-1}ASe_k = S^{-1}Av_k = S^{-1}\lambda_k v_k = \lambda_k S^{-1} v_k = \lambda_k e_k$$

for every k and Be_k is the k'th column vector of B, the matrix B is diagonal with entries λ_k in the diagonal.
Assume now that A is diagonalizable. There exists an invertible matrix S such that $S^{-1}AS = B$ is a diagonal matrix with diagonal entries λ_k. The equation $Be_k = \lambda_k e_k$ means $S^{-1}ASe_k = \lambda_k e_k$ which means after multiplying with S from the left $ASe_k = S\lambda_k e_k = \lambda_k Se_k$. So, $v_k = Se_k$ are eigenvectors with eigenvalues λ_k. Because $\{v_k\}$ is the set of column vectors of S and S is invertible, $\{v_1, \cdots, v_n\}$ is a basis. $\square$

16.4. We don't yet have the answer to the question when an eigenbasis exists, but the theorem shows that the question is important. Here is a sufficient condition:

Theorem: If all eigenvalues of A are different, then an eigenbasis exists.

Proof. If λ_k is an eigenvalue of A, then $A - \lambda_k 1$ has a non-trivial kernel. Let v_k be a non-zero vector in that kernel. Then v_k is an eigenvector of λ_k. To show that $\mathcal{B} = \{v_1, \dots, v_n\}$ are linearly independent, assume that $a_1 v_1 + \cdots + a_n v_n = 0$. Applying $A - \lambda_1 1$ to this from the left gives gives $a_2(\lambda_2 - \lambda_1)v_2 + ... + a_n(\lambda_n - \lambda_1)v_n = 0$. Multiplying this with $A - \lambda_2$ gives $a_3(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)v_3 + ... + a_n(\lambda_n - \lambda_2)(\lambda_n - \lambda_1)v_n = 0$. Multiplying with all $A - \lambda_j$ except λ_k gives $\prod_{j \neq k}(\lambda_j - \lambda_k)a_k v_k = 0$. Since v_k is not zero and all eigenvalues are distinct, $a_k = 0$. This finishes the proof of linear independence. A set of n linear independent vectors in $\mathbb{R}^n$ automatically spans and therefore is a basis. □

16.5. The condition is not necessary: the identity matrix for example is a matrix which is diagonalizable (as it is already diagonal) but which has all eigenvalues 1. The eigenvalues are not distinct. Let us refine the question a bit.

16.6. The **algebraic multiplicity** of an eigenvalue λ of A is the number of times the eigenvalue appears in the list of eigenvalues. More precisely, we can write $p_A(\lambda) = (\lambda_1 - \lambda)^{a_1} \cdots (\lambda_k - \lambda)^{a_k}$, where $\lambda_1, \cdots, \lambda_k$ are all distinct and the multiplicities $a_1, \cdots a_k$ are all positive integers. The integer a_j is called the algebraic multiplicity of the eigenvalue λ_j. For example, the algebraic multiplicity of $\lambda = 1$ in the identity $n \times n$ matrix is n. The statement that all eigenvalues of A are different means that all algebraic multiplicities are 1.

16.7. The **geometric multiplicity** of an eigenvalue λ of A is the dimension of the eigenspace $\ker(A - \lambda 1)$. By definition, both the algebraic and geometric multiplies are integers larger than or equal to 1.

Theorem: geometric multiplicity of λ_k is $\leq$ algebraic multiplicity of λ_k.

Proof. If $v_1, \cdots v_m$ is a basis of $V = \ker(A - \lambda_k)$, we can complement this with a basis $w_1 \cdots, w_{n-m}$ of $V^\perp$ to get a basis of $\mathbb{R}^n$. Let S be the matrix with column vectors $\{v_1, \dots, v_m, w_1, \dots, w_{n-m}\}$. Then $B = S^{-1}AS$ is a partitioned matrix of the form

$$B = \begin{bmatrix} \lambda_k 1 & C \\ 0 & D \end{bmatrix}$$

which has the same characteristic polynomial as A. You have seen in the last homework that the characteristic polynomial of B is the product of the characteristic polynomial of $\lambda_k 1$ and the characteristic polynomial of D. This is $(\lambda_k - \lambda)^m p_D(\lambda)$ showing that $m_{alg}(A) \geq m$. □

16.8. One can do this also by deformation: if $\{v_1, \dots, v_m\}$ is an orthonormal basis of $\ker(A - \lambda_k)$, define $A(t) = A + \sum_{j=1}^m (v_j v_j^T)t^j$ whose eigenvalues to the eigenvector v_j are $\lambda_k + t^j$. If t is positive but small enough, all these eigenvalues are different and also different from any other eigenvalue of A. The characteristic polynomial of $A(t)$ contains therefore the factor $(\lambda - \lambda_k - t)(\lambda - \lambda_k - t^2) \cdots (\lambda - \lambda_k - t^m)$ which in the limit $t \to 0$ gives the factor $(\lambda - \lambda_k)^m$ showing $m_{alg}(\lambda_k) \geq m$. We come back to this perturbation method.

16.9. How do we decide whether a matrix is diagonalizable?

Theorem: A is diagonalizable $\Leftrightarrow m_{\text{alg}}(\lambda) = m_{\text{geom}}(\lambda)$ for all eigenvalues.

Proof. In that case, we have an eigenbasis for A. It is the union of the bases of the individual eigenspaces: $\mathcal{B} = \bigcup_{j=1}^k \mathcal{B}(\ker(A - \lambda_j))$. □

If A is diagonalizable, then any polynomial of A is diagonalizable.

Proof. $S^{-1}f(A)S = f(S^{-1}AS)$ is first shown for polynomials, then by approximation, it follows for any continuous function f.

If A is diagonalizable, then A^T is diagonalizable.

Proof. Assume $S^{-1}AS = B$ is diagonal. Take the transpose to get $S^T A^T (S^{-1})^T = B$ which is $U^{-1}A^T U = B$ with $U = (S^T)^{-1}$.

Examples

16.10. The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is similar to $C = \begin{bmatrix} 6 & 0 & 0 \\ 7 & 4 & 0 \\ 8 & 9 & 1 \end{bmatrix}$. Proof. Both are similar to the same diagonal matrix because both are diagonalizable.

16.11. A rotation dilation matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has the eigenbasis $v_1 = [1, -i]^T$ and $v_2 = [1, i]^T$. The corresponding eigenvalues are $a - ib, a + ib$, where $\alpha = \arg(a + ib)$ is the polar angle to the vector $[a, b]^T$. There is no real eigenbasis if $b \neq 0$. Here is the calculation which shows all this

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} a - ib & 0 \\ 0 & a + ib \end{bmatrix}.$$

16.12. The magic matrix $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ can not be diagonalized because there is no eigenbasis. The rank of A is 1 so that the kernel, the eigenspace to the eigenvalue 0 is only one-dimensional. Similarly, any shear dilation $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ for $b \neq 0$ can not be diagonalized. Note however that $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ can be diagonalized if $a \neq c$.

Homework

This homework is due on Tuesday, 3/12/2019.

Problem 16.1: Assume, A is an invertible matrix with eigenbasis $\mathcal{B}$. Decide whether this is also an eigenbasis for B. If yes, give an argument why (like stating what the eigenvalues are), if no, find a counter example.
a) $B = A^{-1}$. d) $B = e^A$. g) $B_{ij} = A_{ij}^2$.
b) $B = A^T$. e) $B = 1 + A$. h) $B = A^T A$
c) $B = A^3$. f) $B = \text{rref}(A)$. i) $B = A + A^T$

Problem 16.2: Decide from the following matrices, whether there is an eigenbasis. Decide also whether the eigenbasis is real or complex.
a) $\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$. b) $\begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix}$. c) $\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$. d) $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. e) $\begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$.

Problem 16.3: Compute an eigenbasis. If there is an orthonormal eigenbasis, find one. If the matrix is diagonalizable, find B and S such that $S^{-1}AS = B$ is diagonal.
a) $A = \begin{bmatrix} 3 & 17 \\ 10 & 10 \end{bmatrix}$, b) $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & 1 & 1 \end{bmatrix}$, c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Problem 16.4: Group the matrices which are similar!

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 16.5: You should see the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (Make sure you get this, by discussing it.) If you write down the eigensystem or type $Eigensystem[\{\{1,1,1\},\{1,1,1\},\{1,1,1\}\}]$ in mathematica (heaven forbid you should do that!) you obtain an eigenbasis which is not orthonormal. In the next lecture, we will prove that symmetric matrices have an orthonormal eigenbasis.
a) Find an orthonormal eigenbasis to A.
b) Change one 1 to 0 so that there is an eigenbasis but no orthogonal one.
c) Change three entries 1 to 0 in A so that there is no eigenbasis.