LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 11: Determinants

LECTURE

11.1. We have already seen the determinants of 2×2 and 3×3 matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + dhc - gec - hfa - dbi \; .$$

Our goal is to define the determinant for arbitrary matrices and understand the properties of the **determinant functional** det from $M(n,n)$ to $\mathbb{R}$.

11.2. A **permutation** of a set is an invertible map π on this set. It defines a rearrangement of the set. The point x goes to $\pi(x)$. Inductively, one can see that there are $n! = n \cdot (n-1) \cdots 1$ **permutations** of the set $\{1, 2, \dots, n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest. For example, there are $6 = 3 \cdot 2 \cdot 1$ permutations of $\{1, 2, 3\}$. They are $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$. A permutation can be visualized in the form of a **permutation matrix** A. It is a Boolean matrix which has zeros everywhere except at the positions $A_{k\pi(k)}$, where it is 1. An **up-crossing** is a pair $k < l$ such that $\pi(k) < \pi(l)$. When drawing out a permutation matrix, we also call it a **pattern**. The **sign** of a permutation π is defined as $\text{sign}(\pi) = (-1)^u$, where u is the **number of up-crossings** in the pattern of π.

11.3. The **determinant** of a $n \times n$ matrix A is defined by Leibniz as the sum

$$\sum_{\pi} \text{sign}(\pi) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)} \; ,$$

where π is a permutation of $\{1, 2, \dots, n\}$. We see that for $n = 2$, we get two possible permutations, the identity permutation $\pi = (1,2)$ and the flip $\pi = (2,1)$. The determinant of a 2×2 matrix therefore is a sum of two numbers, the product of the diagonal entries minus the product of the side diagonal entries. For $n = 3$, we have 6 permutations and get the **Sarrus formula** stated initially above.

11.4. To organize the summation, one can first choose all the permutations for which $\pi(1) = 1$, then look at all permutations for which $\pi(1) = 2$ etc. This produces the **Laplace expansion**. Let $M(i,j)$ denote the matrix in which the i'th row and j'th column are deleted. Its determinant is called a **minor** of A. For every $1 \le i \le n$:

Theorem: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(M(i,j))$

11.5. This expansion allows to compute the determinant a $n \times n$ matrix by reducing it to a sum of determinants of $(n-1) \times (n-1)$ matrices. It is still not suited to compute the determinant of a 20×20 matrix for example as we would need to sum up $20! = 2432902008176640000$ elements.

11.6. The fastest way to compute determinants for general matrices is by doing a **row reduction**. To understand this, we need the following properties:

Subtracting a row from another row does not change the determinant.
Swapping two rows changes the sign of the determinant.
Scaling a single row by a factor λ multiplies the determinant by λ.

11.7. Let s be the number of swaps and $\lambda_1, \dots, \lambda_k$ the scaling factors which appear when bringing A into row reduced echelon form.

Theorem: $\det(A) = (-1)^s \lambda_1 \cdots \lambda_k \det(\text{rref}(A))$

11.8. We see from this that the determinant "determines" whether a matrix is invertible or not:

Theorem: $\det(A)$ is non-zero if and only if A is invertible.

Here are more properties for $n \times n$ matrices which we prove in class:

$\det(AB) = \det(A)\det(B)$
$\det(A^{-1}) = \det(A)^{-1}$
$\det(SAS^{-1}) = \det(A)$
$\det(A^T) = \det(A)$
$\det(\lambda A) = \lambda^n \det(A)$
$\det(-A) = (-1)^n \det(A)$

11.9. An important thing to keep in mind is that the determinant of a **triangular** matrix is the product of its diagonal elements.

Example: $\det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}) = 20$.

11.10. Another useful fact is that the determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$. Example: $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}) = 2 \cdot 12 = 24$.

Examples

11.11. The determinant of a rotation matrix is either $+1$ or -1: Proof: we know $A^TA = 1$. So, $1 = \det(1) = \det(A^TA) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2$ which forces $\det(A)$ to be either 1 or -1. For a rotation in $\mathbb{R}^2$ the determinant is 1 for a reflection, it is -1. In general, for any rotation the determinant is 1 as we can change the angle of rotation continuously to 0 forcing the determinant to be 1. The determinant depends continuously on the matrix. It can not jump from -1 to 1. Check the proof seminar in Unit 6.

11.12. Find the determinant of the partitioned matrix

$$A = \begin{bmatrix} 3 & 3 & 7 & 3 & 7 & 1 \\ 3 & 5 & 3 & 4 & 1 & 1 \\ 0 & 0 & 4 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} .$$

The determinant is $6 * 2 * 3 = 36$.

11.13. Use row reduction to compute the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

The answer is 8.

11.14. In this example, Laplace expansion is nice. Also row reduction works.

$$A = \begin{bmatrix} 0 & 0 & 0 & 5 & 8 & 0 \\ 3 & 1 & 3 & 4 & 0 & 0 \\ 0 & 5 & 1 & 3 & 2 & 7 \\ 0 & 0 & 7 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \end{bmatrix} .$$

Homework

This homework is due on Thursday, 2/28/2019.

Problem 11.1: Find the determinants of A, B, C: $A = \begin{bmatrix} a^2 & ab \\ ba & b^2 \end{bmatrix}$,

$$B = \begin{bmatrix} 0 & 5 & 7 & 3 & 7 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 3 \\ 3 & 3 & 0 & 0 & 6 & 0 \\ 4 & 2 & 0 & 4 & 0 & 0 \\ 5 & 3 & 2 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 4 & 0 \\ 7 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}$$

Problem 11.2: Is the following determinant positive, zero or negative? (no technology!)

$$\begin{bmatrix} 22 & 100^9 & 7 & -6 & 3 & 1 \\ 100^9 & 22 & 2 & 2 & 2 & 2 \\ 6 & 4 & 22 & 1 & 100^9 & -1 \\ 2 & 2 & 100^9 & 22 & -5 & 9 \\ 9 & 1 & -1 & 100^9 & 22 & 2 \\ 7 & 4 & -1 & 2 & 4 & 100^9 \end{bmatrix}.$$

Problem 11.3: a) Use the Leibniz definition of determinants to show that the **partitioned matrix** satisfies $\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det(A)\det(B)$.

b) Assume now that A, B are $n \times n$ matrices. Can you find a formula for $\det \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$? (It will depend on n.)

c) Show that number of up-crossings of a pattern is the same if the pattern is transposed and that therefore $\det(A^T) = \det(A)$.

Problem 11.4: Find the determinant of the matrix $A_{ij} = 2^{ij}$ for $i, j \leq 4$. It is $\begin{bmatrix} 2 & 4 & 8 & 16 \\ 4 & 16 & 64 & 256 \\ 8 & 64 & 512 & 4096 \\ 16 & 256 & 4096 & 65536 \end{bmatrix}$. First scale some rows to make the computation more manageable.

Problem 11.5: Find a formula for the determinant of the $n \times n$ matrix $L(n)$ which has 2 in the diagonal and 1 in the side diagonals and 0 everywhere else. Compute first $L(2), L(3), L(4)$, then $L(5) = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Now, you see a pattern. Prove it by induction.