LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 9: SU(2)

Seminar

9.1. We have seen this week that the set of orthogonal $n \times n$ matrices forms a group. This means that one can multiply two such matrices AB and still have an orthogonal matrix. Also, the inverse of an orthogonal matrix is orthogonal. The group of orthogonal $n \times n$ matrices is called $O(n)$. We will talk about determinants later in this course but there is a subgroup $SO(n)$ consisting of all orthogonal matrices which have determinant 1. It is a subgroup because $\det(AB) = \det(A)\det(B)$ and $\det(A^T) = \det(A)$. If you have time, verify these identities in the case of 2×2 matrices or 3×3 matrices at the end of this seminar.

9.2. The class of 2×2 matrices with determinant 1 forms a group called $SL(2, R)$. It is called the **special linear group**. We can say that the intersection of $SL(2, R)$ with $O(2)$ is $SO(2)$. It is the special orthogonal group or rotation group in two dimensions.

> **Problem A:** a) Write down the general matrix in the form $SO(2)$.
> b) Write down the 2×2 matrix in $O(2)$ which is not in $SO(2)$?

> **Problem B:** Can you find a continuous path $A(t)$ of matrices, where $A(0)$ is an orthogonal 2×2 matrix with determinant 1 and $A(1)$ is an orthogonal 2×2 matrix with determinant -1? If yes, give a path connecting $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. If none should exist, prove why such a deformation can not exist. What about 3×3 matrices? Can one find a path $A(t)$ in $O(3)$ such that $A(0) = 1$ and $A(1)$ is a reflection at the xy-plane?

9.3. In this course, we again use complex numbers. Let us just recall some things: the set of complex numbers $z = a + ib$ is a linear space and called $\mathbb{C}$. It is also an **algebra** because we can not only add but also multiply. The algebra $\mathbb{C}$ is isomorphic to the space of **rotation dilation matrices**. Indeed, if $z = a + ib$, then we can write $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so that $a1 + bi$ is a rotation dilation matrix. We can say that the rotation-dilation group is a **representation** of the complex algebra.

9.4. Let us look now at complex 2×2 matrices. The same computation as above shows that $\det(AB) = \det(A)\det(B)$. The matrices with determinant 1 are called $SL(2, \mathbb{C})$. They form a group too.

Problem C: Why is $SL(2, \mathbb{C})$ a group? How does one invert a general matrix in $SL(2, \mathbb{C})$?

9.5. As for real matrices, we can look at the transpose matrix A^T. In the complex, there is a more natural involution on matrices, which is called the **adjoint** $A^* = \overline{A}^T$. To get the adjoint of a matrix, one both transposes and conjugates it. One reason why one takes also the conjugate is that then the inner product $A \cdot B = \text{tr}(A^*B)$ behaves like an inner product and especially has the property that $\text{tr}(A^*A) = \sum_{i,j} |A_{ij}|^2$ is non-zero for nonzero matrices.

9.6. The analogue of orthogonal matrices are the **unitary matrices.** A matrix is called **unitary** if $U^*U = 1$. The class of unitary 2×2 matrices is denoted $U(2)$. It has a subgroup $SU(2)$ of unitary transformations with determinant 1. It is called the **special unitary group**.

9.7. So, $SU(2)$ consists of all matrices

$$A = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

with $\det(A) = |z|^2 + |w|^2 = 1$.

Problem C: Write $z = a + ib$ and $w = c + id$ and verify that $SU(2)$ is the **three dimensional sphere** $a^2 + b^2 + c^2 + d^2 = 1$ in $\mathbb{R}^4$.

9.8. We have mentioned the three dimensional sphere a couple of times in Math 22a and seen that the volume is $2\pi^2$. Every compact simply connected three dimensional manifold is topologically a 3-sphere. Very interesting is that S^3 is besides the circle the only unit sphere in a Euclidean space which is also a Lie group.

9.9. We have seen quaternions a couple of times already. These are numbers of the form $a+bi+cj+dk$. The norm of a quaternion was $\sqrt{a^2 + b^2 + c^2 + d^2}$. If a quaternion has norm 1, then it is a **unit quaternion**.

Problem D: Verify that if $a + ib + cj + dk$ is a unit quaternion, then

$$\begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$$

is a matrix in $SU(2)$.

9.10.

Problem E: Verify the determinant identity $\det(AB) = \det(A)\det(B)$ for 2×2 matrices. Either do it by hand or run the following Mathematica code to verify it. While you are at it, verify it also for 3×3 matrices.

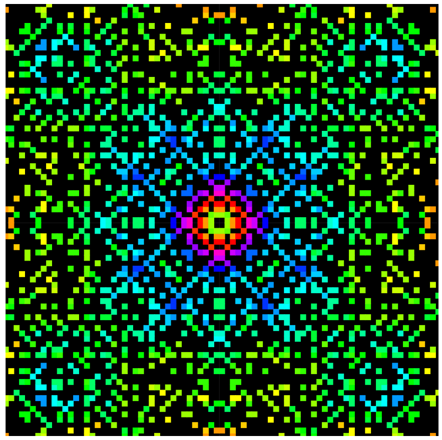

FIGURE 1. We see a few Hurwitz primes on the slice $a = 1/2, b = 1/2$ in the space of integer quaternions (a, b, c, d). For this illustration we take primes of the form $(a, b, c, d) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + (0, 0, k, l)$. This is a prime if and only if $1+k+k^2+l+l^2$ is a **rational prime** (usual prime). In the picture, a Hurwitz prime is colored according how many primes there are close by.

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A={{a,b},{c,d}};   B={{p,q},{r,s}};
Simplify[Det[A.B] == Det[A] Det[B]]
Det[Transpose[A]]==Det[A]
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9.11. Finally, we should mention that the group $SU(2)$ is of great interest in physics as it is the gauge group of the **electroweak interaction**.

FIGURE 2. Sheldon Glashow, Abdus Salam and Steven Weinberg won the Nobel Prize for electroweak unification in 1979.

Homework

Exercises A)-E) are done in the seminar. This homework is due on Tuesday:

Problem 9.1 Verify that if $X = a + ib + jc + kd$ is a quaternion, then the corresponding matrix

$$Q(X) = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$$

has the property that $Q(XY) = Q(X)Q(Y)$. Either do it by hand or look up the Mathematica code presented in unit 40 of Math 22a.

Problem 9.2 Assuming the property $\det(AB) = \det(A)\det(B)$ for 2×2 matrices, use the previous problem to see that if $N(X) = N(a+ib+jc+kd) = a^2 + b^2 + c^2 + d^2$, then $N(XY) = N(X)N(Y)$.
P.S. This compatibility of the quaternions with multiplication makes $\mathbb{H}$ a **division algebra**. One can divide X/Y by forming $X\overline{Y}/|Y|^2$ where $\overline{a + ib + jc + kd} = a - ib - jc - kd$.

Problem 9.3 a) The **Lagrange four square theorem** assures that every positive integer can be written as a sum of four integer squares. How does the previous identity assure that the theorem is proven if one has verified it for primes?
b) Find the smallest integer which can not be written as a sum of 3 squares.
c) Find the smallest integer which can not be written as a sum of 4 non-negative cubes.

Problem 9.4 a) Find a quaternion prime of norm 17.
b) Use the Quaternion identity $Q(XY) = Q(X)Q(Y)$ to find a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = 229 * 179 = 40991$.

Problem 9.5 a) The **Hurwitz quaternions** in $\mathbb{Q}$ are the analog of integers in $\mathbb{R}$ They are either quaternon numbers of the form $a + ib + jc + kd$ with integers a, b, c, d or then half integers $(a + ib + jc + kd) + (1/2, 1/2, 1/2, 1/2)$. A Hurwitz quaternion X is called **prime**, if it can not be written as a product of two Hurwitz quaternions with smaller norm. How can you find primes?
b) Write down the 24 Hurwitz quaternions X have norm $N(X) = 1$. Eight of them have integer entries. Sixteen have half integer entries. They are called the unit quaternions. They form a group. Why?