Fourierreihen

reell

2L period. Funkt.

$$\boxed{\text{reell}\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx}$$

$$a_n = \frac{1}{L}\cdot\int_c^{c+2L} f(x)\cdot\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_c^{c+2L} f(x)\cdot\sin\frac{n\pi x}{L}\,dx$$

$$\text{komplex}\quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_c^{c+2L} f(x)\,e^{-in\pi x/L}\,dx$$

spez. $L = \frac{1}{2}$ $c = 0$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} \qquad c_n = \int_0^1 f(x) e^{-2\pi i n x}\,dx$$

$L = \pi$ $c = -L$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx + \frac{a_0}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

zush: $\ulcorner$ $f(x) = \sum_{n=-\infty}^{\infty} c_n\cdot\cos(n\pi x/L) + i c_n \sin(n\pi x/L)$

$$c_n = \underbrace{\frac{1}{2L}\int_c^{c+2L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx}_{\frac{1}{2}a_n} - i\underbrace{\frac{1}{2L}\int_c^{c+2L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx}_{\frac{1}{2}b_n i}$$

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\tfrac{1}{2}(a_n + b_n i)\cos\frac{n\pi x}{L} + \frac{i}{2}(a_n + b_n i)\sin\frac{n\pi x}{L}\right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}a_n \cos\frac{n\pi x}{L} + \frac{1}{2}b_n \sin\frac{n\pi x}{L}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\cos\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right) \lrcorner$$

$$\boxed{\begin{array}{l}\text{Besselsche Ungl.:}\\ f \in C^0(T^1)\\ \sum_{k=-\infty}^{\infty} |\hat f(k)|^2 \le \|f\|^2\end{array}}$$

$\ulcorner$ $0 \le \|f - S_n\|^2$ $\quad S_n = \sum_{k=-n}^{n} \hat f(k) e^{2\pi i n x}$

$$= \langle f, f\rangle - \langle f, S_n\rangle + \langle S_n, S_n\rangle - \langle S_n, f\rangle$$

$$= \|f\|^2 - 2\sum_{k=-n}^{n}|\hat f(k)|^2 + \sum_{k=-n}^{n}|\hat f(k)|^2$$

denn $\langle f, S_n\rangle = \sum_{k=-n}^{n}\langle f, e^{-2\pi i k x}\rangle \overline{\hat f(k)}$

$$= \sum_{k=-n}^{n} |\hat f|^2 = \langle S_n, f\rangle$$

$$\langle S_n, S_n\rangle = \sum_{k,r=-n}^{n} \hat f(k)\overline{\hat f(r)}\langle e^{-2\pi i k x}, e^{-2\pi i r x}\rangle \lrcorner$$

$$\boxed{D_n(t) = \sum_{k=-n}^{n} e^{2\pi i k t} \qquad \text{Dirichlet kern}}$$

$$\boxed{F_n(t) = \frac{1}{n}\sum_{m=0}^{n-1} D_m(t) \qquad \text{Fejer kern}}$$

$$D_n(t) = \frac{\sin(2n+1)\pi t}{\sin \pi t} \qquad F_n(t) = \frac{1}{n}\left(\frac{\sin \pi n t}{\sin \pi t}\right)^2$$

$\ulcorner$ $D_n(t) = e^{-2\pi i n t}\sum_{k=0}^{2n} e^{2\pi i k t} = e^{-2\pi i n t}\,\frac{e^{2\pi i (2n+1)t} - 1}{e^{2\pi i t} - 1}$

$$= \frac{e^{2\pi i (n+1)t} - e^{-2\pi i n t}}{e^{2\pi i t} - 1} = \frac{e^{2\pi i (n+\frac{1}{2})t} - e^{-2\pi i (n+\frac{1}{2})t}}{e^{\pi i t} - e^{-\pi i t}} = \frac{\sin(2n+1)\pi t}{\sin \pi t} \lrcorner$$

$$\boxed{\begin{array}{l}\text{Fourierentwicklung}\\ f \in C^1(T^1) \quad S_n(x) \xrightarrow{glm} f\\ \sup_{0\le x\le 1} |S_n - f| \le \frac{1}{\sqrt{2}\pi}\|f'\|/\sqrt{n}\\ f(x) = \sum_{n=-\infty}^{\infty}\hat f(n) e^{2\pi i n x}\end{array}}$$

$\ulcorner$ $S_n(x) = \sum_{n=-n}^{n}\int_0^1 f(x) e^{-2\pi i k y}\,dy\cdot e^{2\pi i k x}$

$$= \int_0^1 f(x) D_n(x-y)\,dy = \int_{-1/2}^{1/2} f(x+t) D_n(t)\,dt$$

$$S_n(x) - f = \int_{-1/2}^{1/2}\left(f(x+t) - f(x)\right) D_n(t)\,dt$$

$$= \int_{-1/2}^{1/2} F_x^+(t) e^{2\pi i n t}\,dt - \int_{-1/2}^{1/2} F_x^- e^{-2\pi i n t}$$

$$= \hat F_x^+(-n) - \hat F_x^-(n)$$

$$F_x^{\pm}(t) = \left[\frac{f(x+t) - f(x)}{\sin \pi t}\right]\frac{e^{\pm\pi i t}}{2i} \qquad \leftarrow \text{nicht Fejer kern!}$$

· stetig weil $F_x^{\pm}(0) = \frac{f'(x)}{2\pi i}$

· periodisch

Besselsche Ungl. $\Rightarrow$ $\left.\begin{array}{l}\sum_{n=-\infty}^{\infty}|\hat F_x^+(-k)|^2 \le \|F_x^+\|^2\\ \sum_{n=-\infty}^{\infty}|\hat F_x^-(k)|^2 \le \|F_x^-\|^2\end{array}\right\} \Rightarrow \begin{array}{l}\lim_{n\to\infty}\hat F_x^+(-n) = 0\\ \lim_{n\to\infty}\hat F_x^-(n) = 0\end{array}\Big\} \Rightarrow \lim_{n\to\infty} S_n(x) - f(x) = 0$

punktweise konv.

$$\boxed{\widehat{f'}(k) = 2\pi i k \hat f(k)}$$

$\ulcorner$ $\int_0^1 f(x) e^{-2\pi i n x}\,dx \overset{\text{part. Int.}}{=} -2\pi i n \int_0^1 f(x) e^{-2\pi i n x}\,dx \lrcorner$

$$|S_m(x) - S_n(x)| \le \sum_{n\le|k|\le m} |\hat f(k)| \le \sum_{|k|\ge n} |\hat f(k)| = \sum_{|k|\ge n} |\widehat{f'}(k)|\frac{1}{2\pi k} \overset{\text{Schwarz}}{\le} \left(\sum_{|k|\ge n}|\widehat{f'}(k)|^2\right)^{1/2}\left(\sum_{|k|\ge n}\frac{1}{4\pi^2 k^2}\right)^{1/2}$$

$$= \|f'\|\cdot\frac{1}{\pi\sqrt{2n}} \Rightarrow |f - S_n| \le \|f'\|\cdot\frac{1}{\pi\sqrt{2n}}$$

$$\le \frac{1}{2n\pi^2} \quad \text{denn} \sum_{|k|\ge n}\frac{1}{2k^2} = \sum_{k\ge n}\frac{1}{k^2} < \int_n^{\infty}\frac{1}{x^2}\,dx = \frac{1}{n} \lrcorner$$

$$\boxed{\text{Parseval:}\quad \|f\|^2 = \sum_{k=-\infty}^{\infty}|\hat f(k)|^2 \quad \forall f \in C^1(T^1)}$$

$$\|S_n - f\| = \|f\|^2 - \sum_{k=-n}^{n}|\hat f(k)|^2 \longrightarrow 0$$

$$\boxed{\wedge: C^1(T^1) \longrightarrow l_2 \quad \text{injektiv}}$$

$$\boxed{\begin{array}{l}f \in C^k(T^1) \Rightarrow \sum_{k=-\infty}^{\infty}|\hat f(n)|^2 n^{2k} < \infty\\ \text{Schnelligkeit d. Konvergenz}\end{array}}$$

$$\sum |2\pi i n|^{2k}|\hat f(n)|^2 \le \left\|\frac{d^k}{dx^k}f\right\|^2 < \infty$$

$$\boxed{\begin{array}{l}\text{Satz von Fejer}\\ f \in C^0(T^1) \quad \frac{1}{n}\sum_{m=0}^{n-1} S_m \xrightarrow{glm} f\\ f = \lim_{n\to\infty}\sigma_n \qquad \text{Cesaro}\end{array}}$$

$\ulcorner$ $\sigma_n = \int_{-1/2}^{1/2} f(x+t) F_n(t)$

$$\sigma_n - f = \int_{-1/2}^{1/2}\left(f(x+t) - f(x)\right) F_n(t)\,dt$$

$$= \underbrace{\int_{|t|<\delta}\left(f(x+t) - f(x)\right) F_n(t)\,dt}_{I} + \underbrace{\int_{\delta\le|t|\le 1/2}\left(f(x+t) - f(x)\right) F_n(t)\,dt}_{II}$$

$$I = \sup_{x\in[0,1]}\ \sup_{|t|<\delta} |f(x+t) - f(x)| < \varepsilon \quad |t| < \delta.$$

da f glm stetig auf $[0,1]$ kompakt

$$II \le \frac{2\cdot\|f\|_\infty}{n}\cdot\int_{\delta\le|t|\le 1/2}\left(\frac{1}{\sin \pi t}\right)^2 dt$$

$$\le \frac{4\pi\|f\|_\infty}{n}\left(\frac{1}{\sin \pi\delta}\right)^2$$

$$F_n(t) = \frac{1}{n}\sum_{m=0}^{n-1}\frac{\sin(2m+1)\pi t}{\sin \pi t}$$

$$= \frac{1}{n\sin \pi t}\cdot \operatorname{Im}\sum_{m=0}^{n-1} e^{\pi i t(2m+1)}$$

$$= \frac{1}{n\sin \pi t}\ \operatorname{Im}\ \frac{\cos 2\pi n t - 1 + i\sin 2\pi n t}{2i\sin \pi t}$$

$$\boxed{\begin{array}{l}\text{Fourierreihen als spez. Laurentreihen}\\ f(z)\ \text{meromorph mit Periode } 1\\ f(z) = F(e^{2\pi i z}) = F(\xi)\\ F(\xi) = \sum_{n=-\infty}^{\infty} c_n \xi^n \qquad c_n = \frac{1}{2\pi i}\int_{|\xi|=1} F(\xi)\,\xi^{-n-1}\,d\xi\\ f(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z} \qquad c_n = \int_0^1 f(z) e^{-2\pi i n z}\,dz\end{array}}$$

Anwend. $f(x) = x - \frac{1}{2}$ $\quad \hat f(n) = \int (x - \frac{1}{2}) e^{-2\pi i n x}\,dx = \begin{cases} 0 & n = 0\\ \frac{1}{-2\pi i n} & n \ne 0\end{cases}$

$$\|f\|^2 = \int_0^1 (x - \tfrac{1}{2})^2 dx = \sum_{n=-\infty}^{\infty}|\hat f(n)|^2 = \frac{1}{4\pi^2}\sum_{\substack{n=-\infty\\ n\ne 0}}^{\infty}\frac{1}{n^2} = \frac{1}{12} \Rightarrow \sum\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hilfen

$$\boxed{\begin{array}{l}\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))\\ \sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))\\ \sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))\end{array}}$$

wichtige Potenzreihenentw.

$$\sqrt{1\pm x} = 1 \pm \tfrac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 \pm \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \ldots$$

$$\frac{1}{\sqrt{1\pm x}} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9$$

$$\cot x = \frac{1}{x} - \left(\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945}\ldots\right)$$

$$\frac{1}{\cos x} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \ldots \frac{E_n}{(2n)!}x^{2n}$$

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \ldots$$