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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Total:

Unit 41: Final Exam Review Lecture

Problems

Problem 41R.1) (10 points):
The graph G in Figure 1 represents a surface in which all triangles are oriented counterclockwise. We are given a 1-form = vector field F on G.
a) (3 points) Find the line integral of F along the boundary curve oriented counter clockwise.
b) (3 points) Enter the curl dF values in the triangles.
c) (2 points) What is the sum of all curl values?
d) (2 points) Why are the results in a) and c) the same?

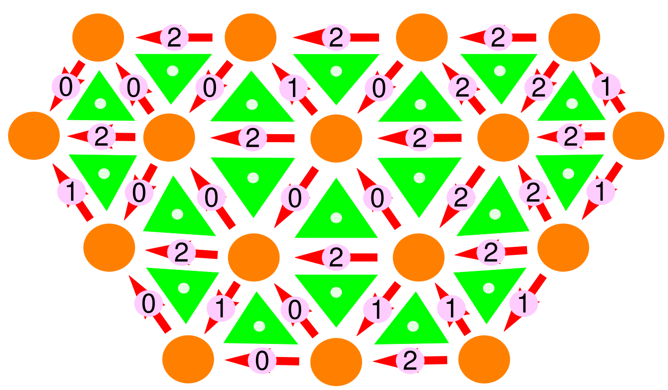

FIGURE 1. A graph analogue to a 2-dimensional region with a 1-form F which models a vector field.

Solution:
a) We compute the line integral by going counter clockwise through the boundary. The line integral is 2.
b) Just compute the curl for each triangles. For example, for the right upper triangle we have the value 1. c) Adding the values up gives 2.
d) This is because of the discrete Stokes theorem (here it can also be seen as the discrete Green theorem).

Problem 41R.2) (10 points) Each question is one point:
a) Assume F is a 1-form on a graph $G = (V, E)$. Define $f = d^*F$. Can you say something about $\sum_{x \in V} f(x)$?
b) The volume $V(S_n)$ of the n-dimensional sphere S_n has the property that $V((S_n) \to ...$
c) Who wrote the book "How to solve?" and who invented differential forms?
d) What is $(1+i)^i$?
e) Green was not only doing mathematics, he had an other profession. Which one? Stokes theorem appeared first in an exam problem. Who was one of the pupils?
f) If B and C are row reductions of the same matrix A. What can you say about the length $|B - C|$?
g) We cited a Harvard professor who invoked the anthropic principle to exclude perpetual motion. Who was this? Also and unrelated: who found first the formula for the volume of the sphere?
h) In the relief below in Figure 2, we see the level curves of some function f. Is the function f a Morse function?
i) True or false? There is a non-zero function $f(x)$ which can be differentiated infinitely many times everywhere which has the property that all derivatives at 0 are 0.
j) What is the name of the lantern which approximates a cylinder but for which the surface area explodes?

Solution:
a) The sum of the divergence values d^*F is zero. One can easily see this from the fact that the $F(x)$ value for an edge x contributes the value $F(x)$ to the vertex at the head of the arrow and the value $-F(x)$ to the vertex at the tail.
b) We have seen the value of the sphere S_n satisfies the recursion $|B_n| = \frac{2\pi}{n}|B_{n-2}|$, and $|S_n| = \frac{2\pi}{n-1}|S_{n-2}|$. This implies that the volume converges to zero.
c) George Polya. He was from 1914-1940 at ETH and then at Stanford university.
Élie Cartan, who was the father of Henry Cartan.
d) First write $1+i = \sqrt{2}e^{i\pi/4}$ so that $\log(1+i) = \log(\sqrt{2}) + i\pi/4 = \log(2)/2 + i\pi/4$. This is because $(1,1)$ has the polar coordinates $(r,\theta) = (\sqrt{2}, \pi/4)$ and $z = re^{i\theta} = x + iy = r\cos(\theta) + ir\sin(\theta)$ is the Euler formula. Now $(1+i)^i = e^{i\log(1+i)} = e^{i\log(2)/2 - \pi/4} = e^{-\pi/4}(\cos(\log(2)/2) + i\sin(\log(2)/2)$. e) Green was a miller. Maxwell took the "Smith's Prize exam" in 1854. It contained Stokes theorem as problem 8.
f) The matrix $B - C$ is the zero matrix. The length of the zero matrix is zero.
g) Benjamin Peirce, Archimedes.
h) It is not a Morse function because there is a critical point which has $D = 0$ (there are three lines crossing) which by the Morse Lemma is not possible.
i) Yes, we have seen that in lecture 18. $f(x) = e^{-1/x}$ has this property.
j) Schwarz lantern.

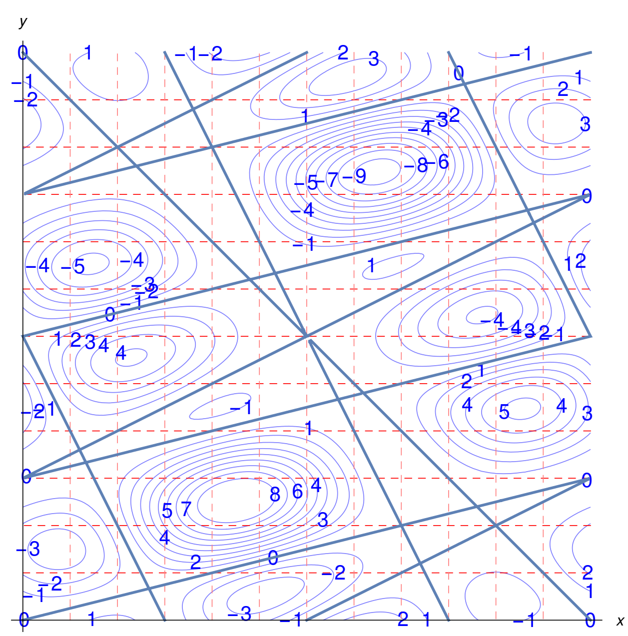

FIGURE 2. Contour map of some function $f(x,y)$.

Problem 41E.3) (10 points) Each problem is 1 point:
a) What is the value of the Hessian determinant $D = \det(d^2f(x))$ at a critical point x of a Morse function f?
b) What is the curl of a vector field F which is conservative?
c) Take the unit sphere and drill a hole of radius $1/10$ from the surface to the center. Is the solid simply connected?
d) One of the three following identities is not defined. F are vector fields in $\mathbb{R}^3$. Which one?
A) $\text{curl}(\text{curl}(F))$,
B) $\text{grad}(\text{div}(F))$,
C) $\text{div}(\text{div}(F))$.
e) Which of the following three vector fields can not be the curl of an other vector field?
A) $F = [x, y, z]$,
B) $F = [y, z, x]$,
C) $F = [z, x, y]$.
f) Which of the following vector fields is a gradient field?
A) $F = [x, y, z]$,
B) $F = [y, z, x]$,
C) $F = [z, x, y]$?
g) What is the exterior derivative dF if $F = xdy + zdz + xdx$?
h) Is it true that $\text{curl}(F)$ is always perpendicular to F?
i) Is there a differentiable function which violates the Fubini theorem? Is there a differentiable function which violates the Clairaut theorem?
j) What is the name of the partial differential equation $\text{curl}(E) = -B_t$?

Solution:
a) D is not zero.
b) It is constant 0 everywhere.
c) Yes,
d) The third one is not defined.
e) Compute the divergence. The first one has divergence different from zero. It can not be the curl of an other vector field.
f) Only 1 is a gradient field. The curl of the others are non-zero.
g) $dxdy$.
h) no.
i) Clairaut j) Maxwell equation

Problem 41R.4) (10 points):
a) Find the equation $ax + by + cz = d$ of the plane which contains both the line $r(t) = [2 - t, t + 1, 3t]$ as well as the point $P = (3, 5, 1)$.
b) What is the distance from P to the line?

Solution:
a) The line contains the point $Q = (2,1,0)$. So, the plane contains also the vector $P - Q = [1,4,1]$. The equation of the plane can now be obtained by taking the cross product of $v = [-1,1,3]$ with $w = [1,4,1]$ which is $[-11,4,-5]$. The equation of the plane is therefore $-11x + 4y - 5z = d$. The constant d can be obtained by plugging in the point $(2,1,0)$ which is -18. The equation of the plane is $-11x + 4y - 5z = -18$.
b) We have already computed $v \times w$. The distance is the area $|v \times w|$ divided by the base length $|v|$. It is $\sqrt{11^2 + 4^2 + 5^2}/\sqrt{1+1+9} = \sqrt{162}/\sqrt{11}$.

Problem 41R.5) (10 points):
a) Find the critical points of the function $f(x,y) = xy + x^2 + 2x$ and classify them using the second derivative test.
b) Does f have a global mazimum or minimum?

Solution:
a) The point is $(0,-2)$. The determinant D is -1, so that this is a saddle.
b) No, putting $x = 1$ gives the function $f(1,y) = y$, which is unbounded.

Problem 41R.6) (10 points):
Use the Lagrange method to find the maximum of xyz under the constraint $x + y + z - yz = 1$.

Solution:
$(1/4, 1/2, 1/2)$ is the maximum. There are more critical points: $(1,0,0)$,
$(0,1,0)$, $(0,0,1)$,
$(0,1,1)$.

Problem 41R.7) (10 points):
The surface $f(x,y,z,w) = x^2 + y^2 + z^2 - w^2 = 1$ is called a **hyper-hyperboloid**. a) Find the tangent plane at the point $(1,0,1,1)$.
b) The tangent plane has the form $ax + by + cz + dw = e$. Parametrize this plane

Solution:
a) The gradient is $\nabla f = (2x, 2y, 2z, -2w)$. At the point $P = (1, 0, 1, 1)$ this is $(2, 0, 2, -2)$. The equation of the tangent plane is $2x + 2z - 2w = 2$.
b) We have to find a point and three vectors in that plane. The easiest is to find 3 more points in that plane like $A = (1, 0, 0, 0), B = (0, 0, 1, 0), C = (0, 0, 0, -1)$, then take $P + t_1(A - P) + t_2(B - P) + t_3(C - P)$. It is the same as in three dimensions we just have now three parameters.

Problem 41R.8) (10 points):
Estimate the cube root of $1001 * 999^2$ using a quadratic approximation of $f(x, y) = (xy^2)^{1/3}$ at a suitable point.

Solution:
The point is $(1000, 1000)$. We have $f_x(1000, 1000) = 1/3$ and $f_y(1000, 1000) = 2/3$. The second derivatives give more work. We have $f_{xx}(1000, 1000) = f_{yy}(1000, 1000) = -1/4500$ and $f_{xy}(1000, 1000) = 1/4500$. The quadratic approximation is

$$Q(x, y) = 1000+(1/3)(x-1000)+(2/3)(y-1000)+(1/4500)(x-1000)^2/2-(1/4500)(y-1000)^2/2+(1/4500)($$

Now $Q(1001, 999) = 4498499/4500$ which is 999.66644444..... The actual result $999^{2/3}1001^{1/3}$ is 999.666222271489739309879757029059 46.

Problem 41R.9) (10 points):
a) We live in 22 dimensional space and observe a planet moving along a path $r(t)$ experiencing the acceleration $r''(t) = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. The planet is initially at the point $r(0) = [10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ and has zero initial velocity. Were is it at time $t = 1$?
b) What is the curvature $|T'(0)|/|r'(0)|$ at $t = 0$?

Solution:
a) It looks like a tough problem as the dimension is high but it is probably the easiest here: just integrate to get

$$r'(t) = [t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t, t] + C$$

where C is a constant vector. As the initial velocity is zero, $C = 0$. Now integrate again to get

$$r(t) = t^2[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]/2 + C$$

where again $C = [10, 0, 0, 0...0]$. So

$$r(1) = [21, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]/2 .$$

b) We have $T(t) = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]/\sqrt{22}$. Therefore $T'(t) = 0$ and the curvature is zero. Actually, the curve is a line and lines have zero curvature.

Problem 41R.10) (10 points):
a) Integrate the function $f(x, y) = y$ over the region given in polar coordinates given as $0 \leq r \leq \theta$.
b) Integrate the function $f(x, y, z) = 3 + x + y$ over the solid given in spherical coordinates as $0 \leq \rho \leq \phi$.

Solution:
The choice of coordinate system is already given. We have still to include the distortion factors:
a) The integral is $\int_0^{2\pi} \int_0^{\theta} r \sin(\theta) r \; dr d\theta$. which gives $4\pi - \frac{8\pi^3}{3}$.
b) The integral is

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\phi} (3 + \rho\cos(\theta)\sin(\phi) + \rho\sin(\theta)\sin(\phi))\rho^2 \sin(\phi) \; d\rho d\phi d\theta \; .$$

Integrating $\cos(\theta)$ and $\sin(\theta)$ over the interval $[0, 2\pi]$ gives zero. So, only $\int_0^{2\pi} \int_0^{\pi} \int_0^{\phi} 3\rho^2 \sin(\phi) \; d\rho d\phi d\theta$ survives. The answer is $2\pi \int_0^{\pi} 3\phi^3 \sin(\phi)/3 \; d\phi$. Use integration by parts to get $2\pi^2(\pi^2 - 6)$.

Problem 41R.11) (10 points):
a) What is the volume of the solid $G : x^4 + y^4 - 1 < z < 1 + 2x^2 + 2y^4$, $|x|^2 < 1, |y|^2 < 1$?
b) What is the average height $\iiint_G z \; dV / \iint_G 1 \; dV$?

Solution:
a) This is a simple "burger" integral, where we integrate from the lower floor $x^4 + y^4 - 1$ to the upper floor $1 + 2x^2 + 2y^4$ over the region R which is a square $[-1, 1] \times [-1, 1]$. The integral is

$$\int_{-1}^{1} \int_{-1}^{1} (1 + 2x^2 + 2y^4) - (x^4 + y^4 - 1)\, dx = 32/3 .$$

b) The upper integral is

$$\iiint_G z\, dV = \int_{-1}^{1} \int_{-1}^{1} \int_{x^4+y^4-1}^{1+2x^2+2y^4} z\, dzdxdy$$

which means integrating $(1+2x^2+2y^4)^2/2 - (x^4+y^4-1)^2/2$ over the square. The result is 1984/225. The final result is $(1984/225)/(32/3) = 62/75$.

Problem 41R.12) (10 points):
Compute the line integral of the field $F = [y^2 + x, y^5, z^3]$ along the path $r(t) = [t + \sin(t), 0, \sin(4t)]$ from $t = 0$ to $t = \pi$.

Solution:
We can not use the fundamental theorem of line integrals directly as this is not a gradient field. (The curl is not zero). However, we can split up the field and write $F = F_1 + F_2 = [x, y^5, z^3] + [y^2, 0, 0]$. The line integral is $\int_C F_1 \cdot dr + \int_C F_2 \cdot dr$. (Divide and conquer). The first integral can be computed using the fundamental theorem of line integrals as it has the potential $f(x, y, z) = x^2/2 + y^6/6 + z^4/4$. The curve C starts at $A = (0, 0, 0)$ and ends at $B = (\pi, 0, 0)$. The fundamental theorem gives $\int_C F_1 \cdot dr = f(B) - f(A) = \pi^2/2$. The second integral can be computed directly $\int_0^\pi F_2 \cdot dr = \int_0^\pi [0, 0, 0] \cdot r'(t)\, dt = 0$. Since the vector field F_2 was zero on the curve, there was no contribution. The final result is $\pi^2/2$.

Problem 41R.13) (10 points):
Find the line integral of the vector field $F(x, y) = [x^{10} + y, y + \sin(\sin(y))]$ along the triangle $C = ABC$ with $A = (0, 0), B = (2, 0), C = (0, 1)$ in the order $A \to B \to C \to A$.

Solution:
This is a problem for Green's theorem. We have $\text{curl}(F) = -1$. The line integral is equal to the double integral $\iint_G (-1)\, dxdy$ where G is the interior of the triangle. This is -1 times the area of the triangle which is $(-1)1 = -1$.

Problem 41R.14) (10 points):
What is the flux of the vector field $F[x,y,z,w] = [x^3, y^3, z^3, w^3]$ through the sphere $x^2 + y^2 + z^2 + w^2 = 1$ oriented outwards? Remember that we can parametrize the four dimensional ball E as $r(\rho,\phi,\theta_1,\theta_2) = [\rho\cos(\phi)\cos(\theta), \rho\cos(\phi)\sin(\theta), \rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta)]$ with $0 \leq \rho \leq 1, 0 \leq \phi \leq \pi/2, 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi$.

Solution:
We use the divergence theorem. The divergence of F is $\text{div}(F) = 3x^2 + 3y^2 + 3z^2 + 3w^2 = 3\rho^2$. Instead of the flux, we can compute the integral over the interior. We use the integration factor $\rho^3 \sin(2\phi)/2$.

$$\iint_E \text{div}(F)\ dV = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} 3\rho^2 \rho^3 \sin(2\phi)/2\ d\theta_1 d\theta_2 d\phi d\rho\ .$$

The result is $3/6(2\pi)^2/2 = \pi^2$.

Problem 41R.15) (10 points):
What is the flux of the curl of the vector field $F[x,y,z,w] = [xyx, x + y^4wx, -y+x, x*w]$ through the disk surface $r(u,v) = [0,u,v,0]$, $u^2+v^2 \leq 1$.

Solution:
We use **Stokes theorem**. Instead of the flux through the two dimensional surface, we can compute the line integral along the boundary of the surface. The boundary is parametrized by $r(t) = [0, \cos(t), \sin(t), 0]$. Now compute

$$\int_0^{2\pi} F(r(t)) \cdot r'(t)\ dt = \int_0^{2\pi} [0, 0, -\cos(t), 0] \cdot [0, -\sin(t), \cos(t), 0] = \int_0^{2\pi} -\cos^2(t)\ dt = -\pi\ .$$