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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Total:

Unit 41: Final Exam Practice

Problems

Problem 41P.1) (10 points):
On the graph G in Figure 1 we are given a 1-form F on a graph $G = (V, E)$.
a) (3 points) Write the values of the curl dF. As a 2-form it is a function on the set T of triangles.
b) (3 points) Compute the "discrete divergence" d^*F, which is a 0-form, a function on the vertices.
c) (4 points) Find the value of the Laplacian $d^*dF + dd^*F$ and enter the values near the edges in Figure 2.

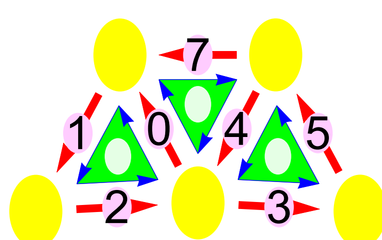


Figure 1. A graph with a 1-Form F. Enter here the result for a) and b).

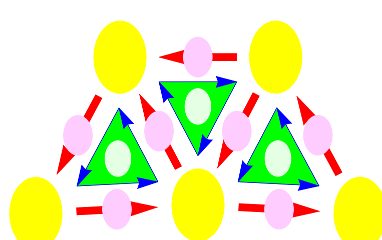
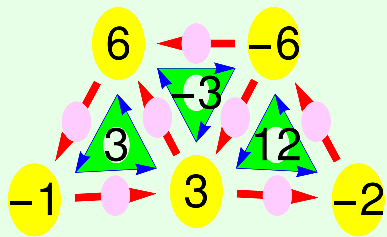
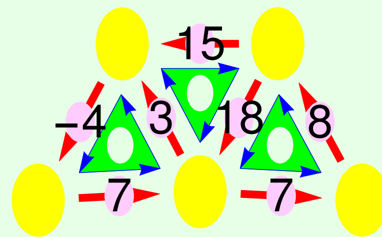


Figure 2. Enter here the result for c).

Solution:
Here are the pictures after computing the derivatives.





Side remark: Greens theorem tells that the sum of dF is the same than the line integral of F along the boundary. This does here not work and the reason is that upper middle triangle is oriented clockwise, not counter clockwise. There is no cancellation going on in the interior.

Problem 41P.2) (10 points) Each question is one point:
a) Who formulated the law of gravity in the form the partial differential equation $\text{div}(F) = 4\pi\sigma$?
b) The expression $5xdxdzdx + 77dydzdy + 3dxdy + 6dydx$ simplifies to ....
c) What value is $\iint_S [x, y, z] \cdot dS$ if S is the unit sphere oriented outwards?
d) What is the distance between the point $(0, 0, 3)$ and the xy-plane?
e) Is it true that if $|r'(t)| = 1$ everywhere, then $r''(t)$ is perpendicular to the velocity $r'(t)$?
f) What is the distortion factor $|dr|$ for the change of coordinates $r(u, v) = [-2v, 3u]$?
g) If $r(u, v)$ parametrizes a surface in $\mathbb{R}^3$, is it true that $r_u \times (r_u \times r_v)$ tangent to the surface?
h) Yes or no: if $(0, 0, 0)$ is a maximum of $f(x, y, z)$ then $f_{xx}(0, 0, 0) < 0$.
i) Write down the quadratic approximation of $1 + x + y + \sin(x^2 - y^2)$?
j) If $S : f(x, y, z) = x^2 + y^2 + z^2 = 1$ is oriented outwards, then the flux of ∇f through S is either negative, zero or positive. Which of the three cases is it?

Solution:
a) Gauss.
b) $-3dxdy$.
c) $3Vol(S) = 4\pi$. This is also the surface area as the field is perpendicular to S and has length 1 there.
d) 3.
e) Yes, if $|r'(t)| = 1$, then $r' \cdot r' = 1$ and differentiation gives $2r' \cdot r'' = 0$.
f) The determinant of the Jacobian which is 6.
g) Yes, we are perpendicular to the normal vector and so parallel.
h) Not necessarily.
i) $1 + x + y + x^2 - y^2$.
j) It is always positive.

Problem 41P.3) (10 points) Each problem is 1 point:
a) Which of the triangles in Figure 3 is integrated over in $\int_0^1 \int_y^1 f(x,y)\, dxdy$?
b) We have seen a counter example for Clairaut's theorem. This function $f(x,y)$ was in C^k but not in C^{k+1}. The integer k indicated how many times we could differentiate f continuously. What was the k?
c) To what group of partial differential equations belongs $\text{div}(E) = 4\pi j + E_t$?
d) Write down the Cauchy-Schwarz inequality.
e) Let G be the first stage of the Menger sponge (with 20 cubes from 27 cubes present). Is it simply connected?
f) Take a exterior derivative of the differential form $F = \sin(xz) dxdy$.
g) Parametrize the surface $x = z^2 - y^3$.
h) Parametrize the curve obtained by intersecting of the ellipsoid $x^2/4 + y^2 + z^2/9 = 1$ with the plane $y = 0$.
i) What surface is given in spherical coordinates as $\sin(\phi)\cos(\theta) = \cos(\phi)$?
j) Write down the general formula for the area of a triangle with vertices $(0,0,0), (a,b,c), (u,v,w)$.

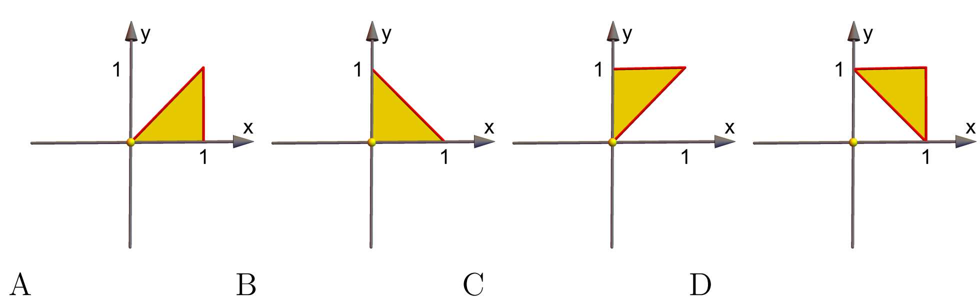


FIGURE 3. Four triangles

Solution:
a) A.
b) $k = 1$.
c) Maxwell equations
d) $|v \cdot w| \leq |v| \cdot |w|$
e) *no*.
f) $x \cos(xz) dzdxdy$.
g) $r(y,z) = [z^2 - y^3, y, z]$.
h) $r(t) = [2\cos(t), 0, 3\sin(t)]$.
i) Multiply both sides with a ρ. This gives the equation $x = z$ which is a plane.
j) $|[a,b,c] \times [u,v,w]| = \sqrt{(bu-av)^2 + (cu-aw)^2 + (cv-bw)^2}/2$.

Problem 41P.4) (10 points):
a) (6 points) Find the equation of the plane which contains the line $r(t) = [1+t, 2+t, 3-t]$ and which is perpendicular to the plane $\Sigma : x+2y-z = 4$.
b) (4 points) What is the angle between the normal vectors of Σ and the plane you just found?

Solution:
a) The plane contains the point $r(0) = [1,2,3]$, the vector $v = [1,1,-1]$ and needs to contain the vector $w = [1,2,-1]$ which is normal to the plane Σ. To find the equation of the plane, we have to find the normal vector $n = [a,b,c]$ to this plane. This n can be obtained as a cross product $n = v \times w = [1,0,1] = [a,b,c]$. The equation of the plane is $ax + by + dz = d$ which is $x + z = d$, where the constant $d = 4$ is obtained by plugging in the point $(1,2,3)$. So, the equation of the plane is $x + z = 4$.
b) We have to find the angle between w and n, which is can be obtained from $\cos(\alpha) = (w \cdot n)/(|w||n|)$ which is 0. The angle between the two vectors is 90 degrees. The two planes are perpendicular to each other. We could have seen this also directly as any plane which contains the normal vector to an other plane is perpendicular to that plane.

Problem 41P.5) (10 points):
a) (8 points) Find the critical points of the function $f(x,y) = \cos(x) + y^5 - 5y$ and classify them using the second derivative test. You can assume that $0 \leq x < 2\pi$.
b) (2 points) Does the function f have a global maximum or a global minimum?

Solution:
a) The gradient is $\nabla f(x,y) = [-\sin(x), 5y^4 - 5]$ which is zero for $x = 0, x = \pi$ and $y = 1$. There are four critical points $(0,1),(0,-1),(\pi,1),(\pi,-1)$. The Hessian matrix H is $H = \begin{bmatrix} -\cos(x) & 0 \\ 0 & 20.y^3 \end{bmatrix}$. It has determinant $D = -\cos(x)y^3$. We have $f_{xx} = -\cos(x)$. We see

point	D	f_{xx}	nature	f value
$(0,-1)$	20	-1	max	5
$(0,1)$	-20	-1	saddle	-3
$(\pi,-1)$	-20	1	saddle	3
$(\pi,1)$	20	1	min	-5

b) No, for $x = 0$ we have the function $y^5 - 5y + 1$ which is unbounded both from above and below.

Problem 41P.6) (10 points):
a) (5 points) Use the Lagrange method to find the maximum of $f(x,y) = y^2 - x$ under the constraint $g(x,y) = x + x^3 - y^2 = 2$.
b) (5 points) The Lagrange equations fail to find the maximum of $f(x,y) = y^2 - x$ under the constraint $g(x,y) = x^3 - y^2 = 0$. Still, the Lagrange theorem still allows you to find the maximum. How?

Solution:
a) The Lagrange equations are

$$\begin{aligned} -1 &= \lambda(1+3x^2) \\ 2y &= \lambda(-2y) \\ x + x^3 - y^2 &= 2 \end{aligned}$$

The first two equations give $2y = (1+3x^2)2y$. This implies $y = 0$ and from the constraint $x = 1$ b) The Lagrange theorem allows allows for the gradient of g to be zero. This also implies that the gradient of f and g are parallel. This happens here at $(0,0)$.

Problem 41P.7) (10 points):
a) (6 points) Find the tangent plane at the point $P = (4,2,1,1)$ of the surface $x^2 - 2y^2 + z^3 + w^2 = 2$.
b) (4 points) Parametrize the line $r(t)$ which passes through P which is perpendicular to the hyper surface at that point. Then find $(r(1) + r(-1))/2$.

Solution:
a) The gradient is $[2x, -4y, 3z^2, 2w]$ which is at the point $(4, 2, 1, 1)$ equal to $n = [8, -8, 3, 2]$. From $8x - 8y + 3z + 2z = e$ we get $e = 21$. The equation of the plane is $8x - 8y + 3z + 2w = 21$.
b) We have $r(t) = [4, 2, 1, 1] + t[8, -8, 3, 2]$. The value $(r(1) + r(-1))/2$ is the point P itself. $P = (4, 2, 1, 1)$.

Problem 41P.8) (10 points):
a) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using linear approximation.
b) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using quadratic approximation.

Solution:
a) The linear approximation is $f(0,0)+f_x(0,0)x+f_y(0,0)y$ as in this case $(x_0, y_0) = (0, 0)$.
b) The Taylor series of $\log(1 + t)$ starts with $t - t^2/2....$ We can just plug in $t = x + 3xy$ and look at quadratic terms. We have $Q(x, y) = x + 3xy - x^2/2$.

Problem 41P.9) (10 points):
a) Lets look at the curve which satisfies the acceleration $r''(t) = [-2\cos(t), -2\sin(t), -2\cos(t), -2\sin(t)]$, has the initial position $[2, 0, 2, 0]$ and initial velocity $[0, 2, 0, 2]$. Find $r(t)$.
b) What is the curvature $|T'(t)|/|r'(t)|$ of $r(t)$ at $t = 0$?

Solution:
a) Integrate twice to get $r(t) = [2\cos(t), 2\sin(t), 2\cos(t), 2\sin(t)]$.
b) $r' = [-2\sin(t), 2\cos(t), -2\sin(t), 2\cos(t)]$ so that $T = [-\sin(t), \cos(t), -\sin(t), \cos(t)]/\sqrt{2}$. $T' = [-\cos(t), -\sin(t), -\cos(t), \sin(t)]/\sqrt{2}$. The curvature is $|T'|/|r'| = 1/(2\sqrt{2})$. The answer is $1/\sqrt{8} = 1/(2\sqrt{2})$. There is a faster to do that: the curve is a circle of radius $2\sqrt{2}$ in four dimensional space, but we could restrict to $x = z, y = w$ which is a two dimensional plane. Now use that the curvature of a circle of radius r is $1/r$.

Problem 41P.10) (10 points):
a) Integrate the function $f(x,y) = x + x^2 - y^2$ over the region $1 < x^2 + y^2 < 4, xy > 0$.
b) Find the **surface area** of

$$r(t,s) = [\cos(t)\sin(s), \sin(t)\sin(s), \cos(s)]$$

$0 \leq t \leq 2\pi, 0 \leq s \leq t/2$.

Solution:
a) The region consists of two quarter rings. One is in the first quadrant, the second is in the third quadrant. Make a picture.
$\int_1^2 \int_0^{\pi/2} r\cos(\theta) + r^2\cos(2\theta)\, d\theta + \int_1^2 \int_{\pi}^{3\pi/2} r\cos(\theta) + r^2\cos(2\theta)\, d\theta = 0$.
b) We know the value of $|r_t \times r_s| = \sin(s)$ already as this is just a sphere of radius 1. We now integrate

$$\int_0^{2\pi} \int_0^{t/2} \sin(s) ds dt = \int_0^{2\pi} 1\, dt = 2\pi\ .$$

Problem 41P.11) (10 points):
Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$

a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2\ dxdydz.$$

b) (3 points) Let F be a vector field

$$F = [x^3, y^3, z^3]$$

Find the flux of F through the boundary surface of E, oriented outwards.

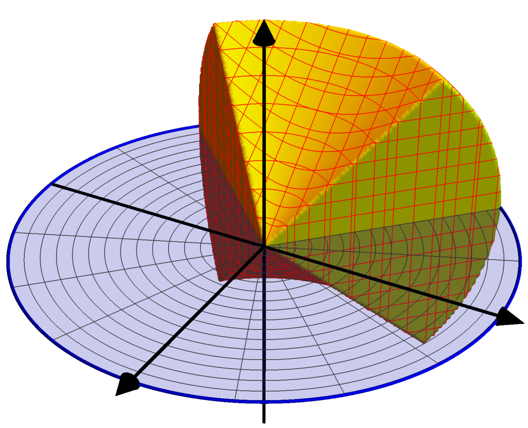


FIGURE 4. The solid in Problem 10.

Solution:
a) The region is best described in **spherical coordinates**. The ϕ angle goes from $\pi/4$ to $3\pi/4$. The θ angle goes from $\pi/4$ to $3\pi/4$. The radius ρ goes from zero to 3. The integral is

$$\int_{\pi/4}^{3\pi/4} \int_{\pi/4}^{3\pi/4} \int_{0}^{3} \rho^2 \rho^2 \sin(\phi) \, d\rho d\phi d\theta \ .$$

The answer is $(\pi/2)(3^5/5)\sqrt{2} = \boxed{243\pi\sqrt{2}/10}$.
b) Since the divergence is $3x^2 + 3y^2 + 3z^2$ the result is just three times the result found in a). It is $\boxed{729\pi\sqrt{2}/10}$.

Problem 41P.12) (10 points):
What is the line integral of the force field $F(x, y, z, w) = [1, 5y^4 + z, 6z^5 + y, 7w^6]^T + [y - w, 0, 0, 0]^T$ along the path $r(t) = [t^3, \sin(6t), \cos(8t), \sin(6t)]$ from $t = 0$ to $t = 2\pi$. Hint. We have written the field by purpose as the sum of two vector fields.

Solution:
Use the fundamental theorem of line integrals for the first part of the vector field and compute the integral directly for the second. But we see that $y - w$ is zero on the path so that the result is just $f(r(2\pi)) - f(r(0))$, where $f = x + y^5 + z^6 + w^7 + yz$. We have $r(2\pi) = [(2\pi)^3, 0, 1, 0]$ and $r(0) = [0, 0, 1, 0]$. The result is $(2\pi)^3$.

Problem 41P.13) (10 points):
Find the area of the region $|x|^{2/5} + |y|^{2/5} \leq 1$. Use an integral theorem.

Solution:
Use the parametrization $r(t) = [\cos(t)^5, \sin(t)^5]$ and the vector field $F(x, y) = [0, x]$. The area is the line integral

$$\int_0^{2\pi} [0, \cos(t)^5] \cdot [5\cos(t)^4 \sin(t), 5\sin(t)^4 \cos(t)]$$

which asks to integrate $\int_0^{2\pi} 5sin(t)^4 \cos(t)^6 \ dt$. This can be done using double angle formulas and gives $15\pi/128$. We integrate $\cos(t)^5 5 - \sin(t)^4 \cos(t)$ which is the same than integrating $(\cos(2t) + 1)\sin(2t)^4/32$ and only the integral $\sin(2t)^4/32$ survives. This gives the result $15\pi/128$.

Problem 41P.14) (10 points):
What is the flux of the vector field $F(x, y, z, w) = [x+\cos(y), y+z^2, 2z, 3w]$ through the boundary of the solid $E : 1 \leq x \leq 3, 3 \leq y \leq 5, 0 \leq z \leq 1, 4 \leq w \leq 8$ oriented outwards?

Solution:
The divergence of F is 7. By the divergence theorem, the result is 7 times the volume of the solid E. Which is $7 * (3-1)(5-3)(1-0)(8-4) = 112$.

Problem 41P.15) (10 points):
Find the **flux** of the curl of the vector field

$$F(x, y, z) = [-z, z + \sin(xyz), x - 3]^T$$

through the **twisted surface** seen in Figure 3 is oriented inwards and parametrized by

$$r(t, s) = [(3 + 2\cos(t))\cos(s), (3 + 2\cos(t))\sin(s), s + 2\sin(t)] \,,$$

where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.

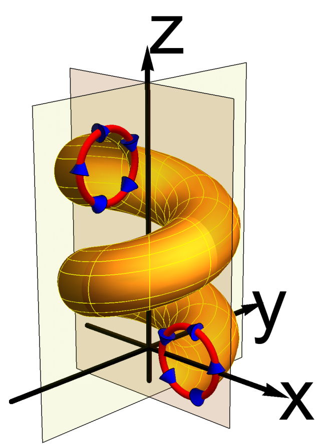


FIGURE 5. The boundary of the surface is made of two circles $r(t, 0)$ and $r(t, 7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).

Solution:
We use Stokes theorem. Instead of computing the flux integral we compute the line integral along the two circles. The first circle is obtained by putting $s = 0$, the second one is obtained by putting $s = 7\pi/2$:

$$r(t) = [3 + 2\cos(t)), 0, 2\sin(t)] \ ,$$

$$r(t) = [0, -3 - 2\cos(t)4\pi, 7\pi/2 + 2\sin(t)] \ .$$

The first line integral $\int_0^{2\pi} [-2\sin(t), 2\sin(t), 2\cos(t)]dt = 8\pi$. The second line integral $\int_0^{2\pi} [7\pi/2 + 2\sin(t), 7\pi/2 + 2\sin(t), -3]dt = 4\pi$ has to be taken negatively. The result is $8\pi - 4\pi = \boxed{4\pi}$.