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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Total:

Unit 41: Final Exam Practice

Problems

Problem 41P.1) (10 points):
On the graph G in Figure 1 we are given a 1-form F on a graph $G = (V, E)$.
a) (3 points) Write the values of the curl dF. As a 2-form it is a function on the set T of triangles.
b) (3 points) Compute the "discrete divergence" d^*F, which is a 0-form, a function on the vertices.
c) (4 points) Find the value of the Laplacian $d^*dF + dd^*F$ and enter the values near the edges in Figure 2.

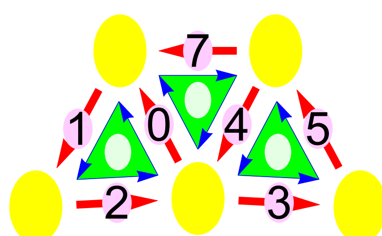


FIGURE 1. A graph with a 1-Form F. Enter here the result for a) and b).

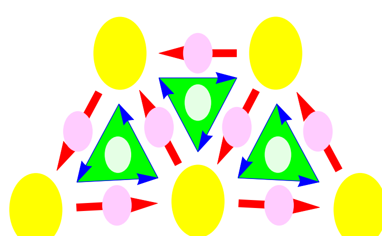
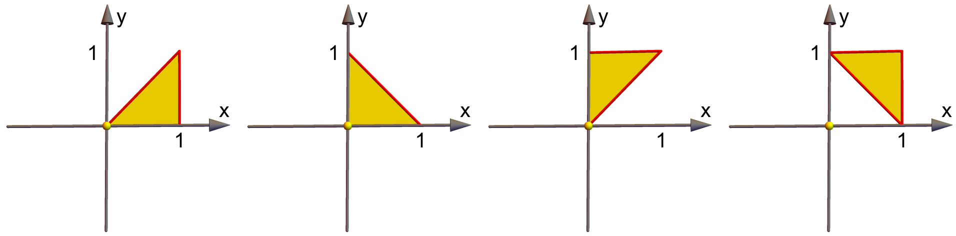


FIGURE 2. Enter here the result for c).

Problem 41P.2) (10 points) Each question is one point:
a) Who formulated the law of gravity in the form the partial differential equation $\text{div}(F) = 4\pi\sigma$?
b) The expression $5xdxdzdx + 77dydzdy + 3dxdy + 6dydx$ simplifies to ....
c) What value is $\iint_S [x, y, z] \cdot dS$ if S is the unit sphere oriented outwards?
d) What is the distance between the point $(0, 0, 3)$ and the xy-plane?
e) Is it true that if $|r'(t)| = 1$ everywhere, then $r''(t)$ is perpendicular to the velocity $r'(t)$?
f) What is the distortion factor $|dr|$ for the change of coordinates $r(u, v) = [-2v, 3u]$?
g) If $r(u, v)$ parametrizes a surface in $\mathbb{R}^3$, is it true that $r_u \times (r_u \times r_v)$ tangent to the surface?
h) Yes or no: if $(0, 0, 0)$ is a maximum of $f(x, y, z)$ then $f_{xx}(0, 0, 0) < 0$.
i) Write down the quadratic approximation of $1 + x + y + \sin(x^2 - y^2)$?
j) If $S : f(x, y, z) = x^2 + y^2 + z^2 = 1$ is oriented outwards, then the flux of ∇f through S is either negative, zero or positive. Which of the three cases is it?

Problem 41P.3) (10 points) Each problem is 1 point:
a) Which of the triangles in Figure 3 is integrated over in $\int_0^1 \int_y^1 f(x, y)\, dxdy$?
b) We have seen a counter example for Clairaut's theorem. This function $f(x, y)$ was in C^k but not in C^{k+1}. The integer k indicated how many times we could differentiate f continuously. What was the k?
c) To what group of partial differential equations belongs $\text{div}(E) = 4\pi j + E_t$?
d) Write down the Cauchy-Schwarz inequality.
e) Let G be the first stage of the Menger sponge (with 20 cubes from 27 cubes present). Is it simply connected?
f) Take a exterior derivative of the differential form $F = \sin(xz)dxdy$.
g) Parametrize the surface $x = z^2 - y^3$.
h) Parametrize the curve obtained by intersecting of the ellipsoid $x^2/4 + y^2 + z^2/9 = 1$ with the plane $y = 0$.
i) What surface is given in spherical coordinates as $\sin(\phi)\cos(\theta) = \cos(\phi)$?
j) Write down the general formula for the area of a triangle with vertices $(0, 0, 0), (a, b, c), (u, v, w)$.

Problem 41P.4) (10 points):
a) (6 points) Find the equation of the plane which contains the line $r(t) = [1+t, 2+t, 3-t]$ and which is perpendicular to the plane $\Sigma : x+2y-z = 4$.
b) (4 points) What is the angle between the normal vectors of Σ and the plane you just found?



A B C D

FIGURE 3. Four triangles

Problem 41P.5) (10 points):
a) (8 points) Find the critical points of the function $f(x, y) = \cos(x) + y^5 - 5y$ and classify them using the second derivative test. You can assume that $0 \le x < 2\pi$.
b) (2 points) Does the function f have a global maximum or a global minimum?

Problem 41P.6) (10 points):
a) (5 points) Use the Lagrange method to find the maximum of $f(x, y) = y^2 - x$ under the constraint $g(x, y) = x + x^3 - y^2 = 2$.
b) (5 points) The Lagrange equations fail to find the maximum of $f(x, y) = y^2 - x$ under the constraint $g(x, y) = x^3 - y^2 = 0$. Still, the Lagrange theorem still allows you to find the maximum. How?

Problem 41P.7) (10 points):
a) (6 points) Find the tangent plane at the point $P = (4, 2, 1, 1)$ of the surface $x^2 - 2y^2 + z^3 + w^2 = 2$.
b) (4 points) Parametrize the line $r(t)$ which passes through P which is perpendicular to the hyper surface at that point. Then find $(r(1) + r(-1))/2$.

Problem 41P.8) (10 points):
a) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using linear approximation.
b) Estimate $f(0.012, 0.023)$ for $f(x, y) = \log(1 + x + 3xy)$ using quadratic approximation.

Problem 41P.9) (10 points):
a) Lets look at the curve which satisfies the acceleration $r''(t) = [-2\cos(t), -2\sin(t), -2\cos(t), -2\sin(t)]$, has the initial position $[2, 0, 2, 0]$ and initial velocity $[0, 2, 0, 2]$. Find $r(t)$.
b) What is the curvature $|T'(t)|/|r'(t)|$ of $r(t)$ at $t = 0$?

Problem 41P.10) (10 points):
a) Integrate the function $f(x,y) = x + x^2 - y^2$ over the region $1 < x^2 + y^2 < 4, xy > 0$.
b) Find the **surface area** of

$$r(t,s) = [\cos(t)\sin(s), \sin(t)\sin(s), \cos(s)]$$

$0 \leq t \leq 2\pi, 0 \leq s \leq t/2$.

Problem 41P.11) (10 points):
Let E be the solid

$$x^2 + y^2 \geq z^2, x^2 + y^2 + z^2 \leq 9, y \geq |x|.$$

a) (7 points) Integrate

$$\iiint_E x^2 + y^2 + z^2 \; dxdydz.$$

b) (3 points) Let F be a vector field

$$F = [x^3, y^3, z^3]$$

Find the flux of F through the boundary surface of E, oriented outwards.

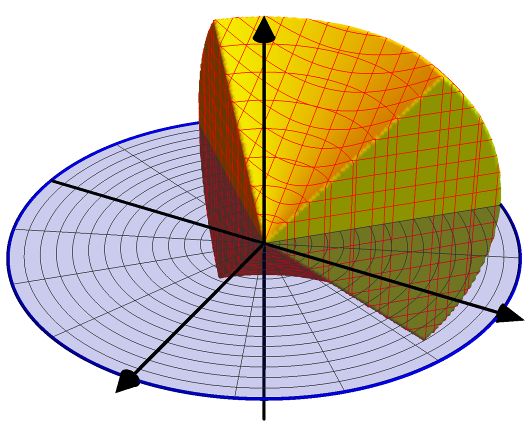


FIGURE 4. The solid in Problem 10.

Problem 41P.12) (10 points):
What is the line integral of the force field $F(x,y,z,w) = [1, 5y^4 + z, 6z^5 + y, 7w^6]^T + [y - w, 0, 0, 0]^T$ along the path $r(t) = [t^3, \sin(6t), \cos(8t), \sin(6t)]$ from $t = 0$ to $t = 2\pi$. Hint. We have written the field by purpose as the sum of two vector fields.

Problem 41P.13) (10 points):
Find the area of the region $|x|^{2/5} + |y|^{2/5} \leq 1$. Use an integral theorem.

Problem 41P.14) (10 points):
What is the flux of the vector field $F(x,y,z,w) = [x+\cos(y), y+z^2, 2z, 3w]$ through the boundary of the solid $E: 1 \leq x \leq 3, 3 \leq y \leq 5, 0 \leq z \leq 1, 4 \leq w \leq 8$ oriented outwards?

Problem 41P.15) (10 points):
Find the **flux** of the curl of the vector field

$$F(x,y,z) = [-z, z + \sin(xyz), x - 3]^T$$

through the **twisted surface** seen in Figure 3 is oriented inwards and parametrized by

$$r(t,s) = [(3 + 2\cos(t))\cos(s), (3 + 2\cos(t))\sin(s), s + 2\sin(t)] \,,$$

where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.

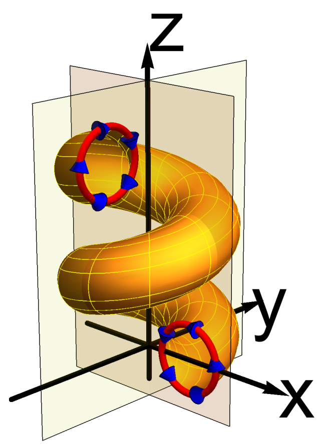


FIGURE 5. The boundary of the surface is made of two circles $r(t,0)$ and $r(t,7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).