LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 41: Keywords for Final (see also Units 13+28)

Discrete Calculus

- $G = (V, E)$ graph with vertex set V and edge set E.
- 0-form: function on V. Discrete scalar function
- 1-form: function on E. Discrete vector field
- 2-form: function on triangles T.
- $d(f) = \text{grad}(f)$ is a function on edges $a- > b$ defined by $f(b) - f(a)$.
- $H = dF = \text{curl}(F)$ is a function on triangles obtained by summing F along the triangle.
- d^*H is a function on edges. Add up the attached triangle values.
- d^*F is a function on vertices. Add up the attached edge values.

New People

- Cartan, Maxwell, Stokes, Green, Gauss, Newton, Maxwell, Kirchhoff, Menger, Koch, Escher, Peirce

Partial Derivatives

- $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ linear approximation
- $Q(x, y) = L(x_0, y_0) + f_{xx}(x - x_0)^2/2 + f_{yy}(y - y_0)^2/2 + f_{xy}(x - x_0)(y - y_0)$.
- use $L(x, y)$ to estimate $f(x, y)$ near $f(x_0, y_0)$. The result is $f(x_0, y_0)+a(x-x_0)+b(y-y_0)$
- tangent plane: $ax + by + cz = d$ with $a = f_x, b = f_y, c = f_z$, $d = ax_0 + by_0 + cz_0$
- estimate $f(x, y)$ by $L(x, y)$ or $Q(x, y)$ near (x_0, y_0)
- $f_{xy} = f_{yx}$ Clairaut's theorem for functions which are in C^2.
- $r_u(u, v)$, $r_v(u, v)$ tangent to surface parameterized by $r(u, v)$

Parametrization

- $r : G \subset \mathbb{R}^m \to \mathbb{R}^n$, dr Jacobian
- $g = dr^T dr$ first fundamental form, $|dr| = \sqrt{g}$ distortion factor.
- $\text{curl}(F)(r(u, v)) \cdot (r_u \times r_v) = F_u \cdot r_v - F_v \cdot r_u$ important formula

Partial Differential Equations

- $f_{xy} = f_{yx}$ Clairaut
- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation

- $f_t + f f_x = f_{xx}$ Burgers equation
- $dF^* = j, dF = 0$, Maxwell equations
- $\text{div}(F) = 4\pi\sigma$, Gravity equation

Gradient

- $\nabla f(x,y) = [f_x, f_y]^T$, $\nabla f(x,y,z) = [f_x, f_y, f_z]^T$, gradient
- $D_v f = \nabla f \cdot v$ directional derivative
- $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r\,'(t)$ chain rule
- $\nabla f(x_0, y_0)$ is orthogonal to the level curve $f(x,y) = c$ containing (x_0, y_0)
- $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ containing (x_0, y_0, z_0)
- $\frac{d}{dt} f(x + tv) = D_v f$ by chain rule
- $(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) = 0$ tangent line
- $(x - x_0) f_x(x_0, y_0, z_0) + (y - y_0) f_y(x_0, y_0, z_0) + (z - z_0) f_z(x_0, y_0, z_0) = 0$ tangent plane
- $D_v f(x_0, y_0)$ is maximal in the $v = \nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$ direction
- $f(x,y)$ increases in the $\nabla f/|\nabla f|$ direction at points which are not critical points
- if $D_v f(x) = 0$ for all v, then $\nabla f(x) = 0$
- $f(x,y,z) = c$ defines $y = g(x,y)$, and $g_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$ implicit diff

Extrema

- $\nabla f(x,y) = [0,0]^T$, critical point
- $D = \det(d^2 f) = f_{xx} f_{yy} - f_{xy}^2$ discriminant.
- Morse: critical point and $D \neq 0$, in 2D looks like $x^2 + y^2, x^2 - y^2, -x^2 - y^2$
- $f(x_0, y_0) \geq f(x,y)$ in a neighborhood of (x_0, y_0) local maximum
- $f(x_0, y_0) \leq f(x,y)$ in a neighborhood of (x_0, y_0) local minimum
- $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, λ Lagrange equations
- $\nabla f(x,y,z) = \lambda \nabla g(x,y,z), g(x,y,z) = c$, λ Lagrange equations
- second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle point**
- $f(x_0, y_0) \geq f(x,y)$ everywhere, global maximum
- $f(x_0, y_0) \leq f(x,y)$ everywhere, global minimum

Double Integrals

- $\iint_R f(x,y)\, dydx$ double integral
- $\int_a^b \int_c^d f(x,y)\, dydx$ integral over rectangle
- $\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx$ bottom-top region
- $\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dxdy$ left-right region
- $\iint_R f(r,\theta) \boxed{r}\, drd\theta$ polar coordinates
- $\iint_R |r_u \times r_v|\, dudv$ surface area
- $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$ Fubini
- $\iint_R \boxed{1}\, dxdy$ area of region R
- $\iint_R f(x,y)\, dxdy$ signed volume of solid bound by graph of f and xy-plane

Triple Integrals

$\iiint_R f(x,y,z)\, dzdydx$ triple integral

$\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx$ integral over rectangular box

$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\, dzdydx$ type I region

$\iiint_R f(r,\theta,z) \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates

$\iiint_R f(\rho,\theta,\phi) \boxed{\rho^2 \sin(\phi)}\ d\rho d\phi d\theta$ integral in spherical coordinates

$\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\, dxdydz$ Fubini

$V = \iiint_E \boxed{1}\, dzdydx$ volume of solid E

$M = \iiint_E \sigma(x,y,z)\, dxdydz$ mass of solid E with density σ

Line Integrals

$F(x,y) = [P(x,y), Q(x,y)]^T$ vector field in the plane

$F(x,y,z) = [P(x,y,z), Q(x,y,z), R(x,y,z)]^T$ vector field in space

$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r\,'(t)\, dt$ line integral

$F(x,y) = \nabla f(x,y)$ gradient field = potential field = conservative field

Fundamental theorem of line integrals

FTL: $F(x,y) = \nabla f(x,y)$, $\int_a^b F(r(t)) \cdot r\,'(t))\, dt = f(r(b)) - f(r(a))$

closed loop property $\int_C F\, dr = 0$, for all closed curves C

always equivalent: closed loop property, path independence and gradient field

mixed derivative test $\text{curl}(F) \neq 0$ assures F is not a gradient field

in simply connected regions: $\text{curl}(F) = 0$ implies that field F is conservative

Conservative field: can not be used for perpetual motion.

Green's Theorem

$F(x,y) = [P,Q]^T$, curl in two dimensions: $\text{curl}(F) = Q_x - P_y$

Green's theorem: C boundary of R, then $\int_C F \cdot dr = \iint_R \text{curl}(F)\, dxdy$

Area computation: Take F with $\text{curl}(F) = Q_x - P_y = 1$ like $F = [-y, 0]^T$ or $F = [0, x]^T$

Green's theorem is useful to compute difficult line integrals or difficult 2D integrals

Flux integrals

$F(x,y,z)$ vector field, $S = r(R)$ parametrized surface

$r_u \times r_v\, dudv = dS$ 2-form on surface

$\int \int_S F \cdot dS = \int \int_S F(r(u,v)) \cdot (r_u \times r_v)\, dudv$ flux integral

Stokes Theorem

$F(x,y,z) = [P,Q,R]^T$, $\text{curl}([P,Q,R]^T) = [R_y - Q_z, P_z - R_x, Q_x - P_y]^T = \nabla \times F$

Stokes's theorem: C boundary of surface S, then $\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot dS$

Stokes theorem allows to compute difficult flux integrals or difficult line integrals

Grad Curl Div

$\nabla = [\partial_x, \partial_y, \partial_z]^T$, $F = \nabla f$, $\text{curl}(F) = \nabla \times F$, $\text{div}(F) = \nabla \cdot F$

$\text{div}(\text{curl}(F)) = 0$ and $\text{curl}(\text{grad}(f)) = 0$

- $\text{div}(\text{grad}(f)) = \Delta f$ Laplacian
- incompressible = divergence free field: $\text{div}(F) = 0$ everywhere. Implies $F = \text{curl}(H)$
- irrotational = $\text{curl}(F) = 0$ everywhere. Implies $F = \text{grad}(f)$

Divergence Theorem

- $\text{div}([P, Q, R]^T) = P_x + Q_y + R_z = \nabla \cdot F$
- divergence theorem: solid E, boundary S then $\iint_S F \cdot dS = \iiint_E \text{div}(F)\, dV$
- the divergence theorem allows to compute difficult flux integrals or difficult 3D integrals

Some topology

- simply connected region D: can deform any closed curve within D to a point
- interior of a region D: points in D for which small neighborhood is still in D
- boundary of curve C: the end points of the curve
- boundary of S points on surface not in the interior of the parameter domain
- boundary of solid G: points in G which are not in the interior of D
- closed surface: a surface without boundary like a sphere
- closed curve: a curve with no boundary like a knot

Some surface parameterizations

- sphere of radius ρ: $r(u, v) = [\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)]^T$
- graph of function $f(x, y)$: $= r(u, v) = [u, v, f(u, v)]^T$
- example: Paraboloid: $r(u, v) = [u, v, u^2 + v^2]^T$.
- plane containing P and vectors u, v: $r(s, t) = P + su + tv$
- surface of revolution: distance $g(z)$ of $z - axis : r(u, v) = [g(v)\cos(u), g(v)\sin(u), v]^T$
- example: Cylinder: $r(u, v) = [\cos(u), \sin(u), v]^T$
- example: Cone: $r(u, v) = [v\cos(u), v\sin(u), v]^T$
- example: Paraboloid: $r(u, v) = [\sqrt{v}\cos(u), \sqrt{v}\sin(u), v]^T$

Integration for integral theorems

- Double and triple integral: $\iint_G f(x, y) dA, \iiint_G f(x, y, z) dV$.
- Line integral: $\int_a^b F(r(t)) \cdot r\,'(t)\, dt$
- Flux integral: $\int\int_S F(r(u, v)) \cdot (r_u \times r_v)\, dudv$

Differential forms

- A k-form is a field, which attaches at every point a multi-linear anti-symmetric map of k variables.
- $F = 5x^3 dydz + 7\sin(y)xdxdz + 3\cos(xy)dxdy$ is an example of a 2-form. In calculus this is identified with a vector field $F = [5x^3, 7\sin(y)x, 3\cos(xy)]$.
- The exterior derivative of a term like $F = Pdxdy$ is $dF = (P_x dx + P_y dy + P_z dz)dxdy = P_z dzdxdy = P_z dxdydz$.
- The general stokes theorem tells $\int_G dF = \int_{\delta G} F$, where δG is the boundary of G.

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