

# LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

## Unit 31: Green's theorem

### LECTURE

**31.1.** For a  $C^1$  vector field  $F = [P, Q]$  in a region  $G \subset \mathbb{R}^2$ , the **curl** is defined as  $\text{curl}(F) = Q_x - P_y$ . Assume the boundary  $C$  of  $G$  oriented so that the region  $G$  is **to the left** (meaning that if  $r(t) = [x(t), y(t)]$  is a parametrization, then the turned velocity  $[-y'(t), x'(t)]$  cuts through  $G$  close to  $r(t)$ ). **Green's theorem** assures that if  $C$  is made of a finite collection of smooth curves, then

$$\textbf{Theorem: } \iint_G \text{curl}(F) \, dxdy = \int_C F(r(t)) \cdot dr(t).$$

**31.2. Proof.** It is enough to prove the theorem for  $F = [0, Q]$  or  $F = [P, 0]$  separately and for regions  $G$  which are both “bottom to top”  $G = B = \{a \leq x \leq b, c(x) \leq y \leq d(x)\}$  and “left to right”  $G = L = \{c \leq y \leq d, a(y) \leq x \leq b(y)\}$ . For  $F = [P, 0]$ , use a bottom to top integral, where the two vertical integrals along  $r(t) = [b, t]$  and  $r(t) = [a, t]$  are zero. The integrals along  $r(t) = [t, c(t)]$  and  $r(t) = [t, d(t)]$  give

$$\int_b^a P(t, c(t)) \, ds - \int_a^b P(t, d(t)) \, ds = \int_a^b \int_{c(t)}^{d(t)} -P_y(t, s) \, dsdt = \iint_G -P_y \, dsdt .$$

For  $F = [Q, 0]$ , use a left to right integral, where the bottom and top integrals are zero and where

$$\int_c^d Q(b(t), t) \, dt - \int_c^d Q(a(t), t) \, dt = \int_c^d \int_{a(s)}^{b(s)} Q_x(t, s) \, dt ds = \iint_G Q_x \, dsdt .$$

In general, write  $F = [0, Q] + [P, 0]$ , use the first computation for  $[P, 0]$  and the second computation for  $[0, Q]$ . In general, cut  $G$  along a small grid so that each part is of both types. When adding the line integrals, only the boundary survives. QED.

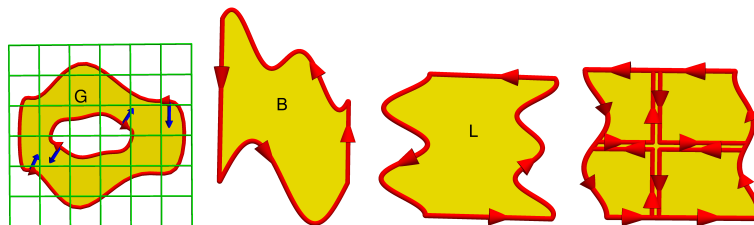


FIGURE 1. To prove Green cut the region into regions which are “bottom to top” and “left to right”. Interior cuts cancel.

**31.3.** To see that we can cut  $G$  into regions of both types, turn the coordinate system first a tiny bit so that no horizontal nor vertical line segments appear at the boundary. This is possible because we assume the boundary to consist of finitely many smooth pieces. Now also use a slightly turned grid to chop up the region into smaller parts. Now we have a situation where each piece has the form  $G = \{(x, y) \mid c(x) \leq y \leq d(x)\} = \{(x, y) \mid a(y) \leq x \leq b(y)\}$ , where  $a, b, c, d$  are piecewise smooth functions.

**31.4.** Green assures:

**Theorem:** If  $F$  is irrotational in  $\mathbb{R}^2$ , then  $F$  is a gradient field.

**31.5.** There are four properties which are equivalent if  $F$  is differentiable in  $\mathbb{R}^2$ : A)  $F$  is a gradient field, B)  $F$  has the closed loop property, C)  $F$  has the path independence property, and D)  $F$  is irrotational. We have seen in the proof seminar that the vortex vector field  $F = [-y, x]/(x^2 + y^2)$  is a counter example to a more general theorem if the field is not differentiable at some point.

## APPLICATIONS

**31.6.** Green's theorem allows to compute **areas**. If  $\text{curl}(F) = 1$  and  $C$  is a curve enclosing a region  $G$ , then  $\text{Area}(G) = \int_C F(r(t)) \cdot r'(t) dt$ . For example, with  $F = [-y, x]/2$ , and  $r(t) = [a \cos(t), b \sin(t)]$ , then  $\int_C F \cdot dr = \int_0^{2\pi} [-b \sin(t), a \cos(t)] \cdot [-a \sin(t), b \cos(t)]/2 dt = \int_0^{2\pi} ab/2 dt = \pi ab$  is the **area of the ellipse**  $x^2/a^2 + y^2/b^2 = 1$ .

**31.7.** What is the area of the region enclosed by  $r(t) = [\cos(t), \sin(t) + \cos(22t)/22]$ ? Take  $F(x, y) = [0, x]$ . The line integral is  $\int_0^{2\pi} [0, \cos(t)] \cdot [-\sin(t), \cos(t) - \sin(22t)] dt = \pi$ .

**31.8.** The **planimeter** is an analogue computer which computes the area of regions. It works because of Green's theorem. The vector  $F(x, y)$  is a unit vector perpendicular to the second leg  $(a, b) \rightarrow (x, y)$  if  $(0, 0) \rightarrow (a, b)$  is the second leg. Given  $(x, y)$  we find  $(a, b)$  by intersecting two circles. The magic is that the curl of  $F$  is constant 1. The following computer assisted computation proves this:

```
s=Solve[{(x-a)^2+(y-b)^2==1,a^2+b^2==1},{a,b}];
{A,B}=First[{a,b}/.s];F={-(y-B),x-A};Simplify[Curl[F,{x,y}]]
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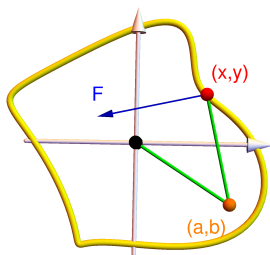


FIGURE 2. The planimeter is an analog computer which allows to compute the area of a region enclosed by a curve.

## EXAMPLES

**31.9. Problem:** Compute  $F(x, y) = [x^2 - 4y^3/3, 8xy^2 + y^5]$  along the boundary of the rectangle  $[0, 1] \times [0, 2]$  oriented counter clockwise. **Solution:** Since  $\text{curl}(F) = Q_x - P_y = 8y^2 + 4y^2 = 12y^2$  we have  $\int_C F \cdot dr = \int_0^1 \int_0^2 12y^2 dy dx = 32$ .

**31.10. Problem:** Find the line integral of the vector field

$$F(x, y) = \begin{bmatrix} x + y \\ 3x + 3y^2 \end{bmatrix}$$

along the boundary  $C$  of the quadratic **Koch island**. The counter clockwise oriented  $C$  encloses the island  $G$  which has 289 unit squares. **Solution:**  $\text{curl}(F) = 2$ , so that  $\iint_G 2dA = 2\text{Area}(G) = 578$ .

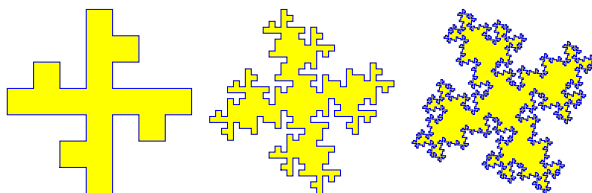


FIGURE 3. Koch islands constructed by a **Lindenmayer system**, a recursive grammar. It starts with  $F + F + F + F$  and recursion  $F \rightarrow F - F + F + FFF - F - F + F$ . [F=“moving forward by 1”, + = “turn by 90 degrees”, - = “turn by (-90) degrees”.]

## HOMEWORK

**Problem 31.1:** Calculate the line integral  $\int_C F \cdot dr$  with  $F = [-22y + 3x^2 \sin(y) + 2222 \sin(x^6), x^3 \cos(y) + 2342y^{22 \sin(y)}]^T$  along a triangle  $C$  which traverses the vertices  $(0, 0)$ ,  $(7, 0)$  and  $(7, 11)$  back to  $(0, 0)$  in this order.

**Problem 31.2:** A classical problem asks to compute the area of the region bounded by the **hypocycloid**

$$r(t) = [4 \cos^3(t), 4 \sin^3(t)], 0 \leq t \leq 2\pi.$$

We can not do that directly so easily. Guess which theorem to use, then use it!

**Problem 32.3:** Find  $\int_C [\sin(\sqrt{1+x^3}), 7x] \cdot dr$ , where  $C$  is the boundary of the region  $K(n)$ . You see in the picture  $K(0), K(1), K(2), K(3), K(4)$ . The first  $K(0)$  is an equilateral triangle of length 1. The second  $K(1)$  is  $K(0)$  with 3 equilateral triangles of length  $1/3$  added.  $K(2)$  is  $K(1)$  with  $3 * 4^1$  equilateral triangles of length  $1/9$  added.  $K(3)$  is  $K(2)$  with  $3 * 4^2$  of length  $1/27$  added and  $K(4)$  is  $K(3)$  with  $3 * 4^3$  triangles of length  $1/81$  added. What is the line integral in the Koch Snowflake limit  $K = K(\infty)$ ? The curve  $K$  is a **fractal** of dimension  $\log(4)/\log(3) = 1.26 \dots$ .

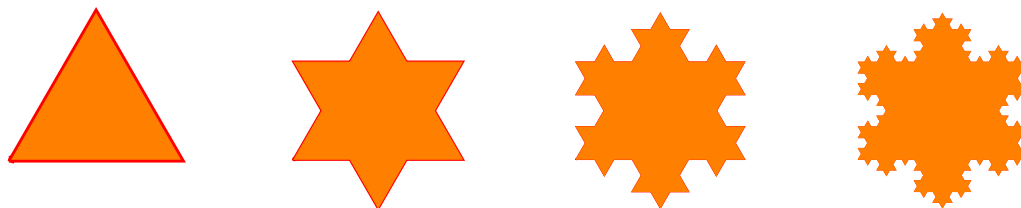


FIGURE 4. The first 4 approximations of the Koch curve.

**Problem 32.4:** Given the scalar function  $f(x, y) = x^5 + xy^4$ , compute the line integral of

$$F(x, y) = [5y - 3y^2, -6xy + y^4] + \nabla(f)$$

along the boundary of the **Monster region** given in the picture. There are four boundary curves, oriented as shown in the picture: a large ellipse of area 16, two circles of area 1 and 2 as well as a small ellipse (the mouth) of area 3. “Mike” from **Monsters, Inc.** warns you about orientations!

**Problem 32.5:** Let  $C$  be the boundary curve of the white Yang part of the Yin-Yang symbol in the disc of radius 6. You can see in the image that the curve  $C$  has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$F(x, y) = [-y + \sin(e^x), x]^T$$

along  $C$ . There are three separate line integrals.

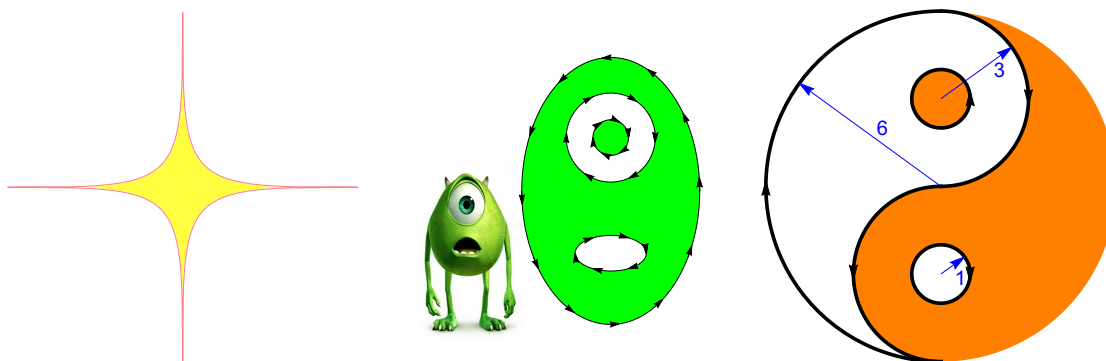


FIGURE 5. Hypocycloid, Monster and Yin-Yang