LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 28: Keywords for Second Hourly

Partial Derivatives

- $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ partial derivative
- $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ linear approximation
- $Q(x,y) = L(x_0,y_0) + f_{xx}(x-x_0)^2/2 + f_{yy}(y-y_0)^2/2 + f_{xy}(x-x_0)(y-y_0)$ quadratic
- $L(x,y)$ estimates $f(x,y)$ near $f(x_0,y_0)$. The result is $f(x_0,y_0) + a(x-x_0) + b(y-y_0)$
- tangent line: $ax+by=d$ with $a = f_x(x_0,y_0), b = f_y(x_0,y_0)$, $d = ax_0 + by_0$
- tangent plane: $ax+by+cz=d$ with $a = f_x, b = f_y, c = f_z$, $d = ax_0 + by_0 + cz_0$
- estimate $f(x,y,z)$ by $L(x,y,z)$ near (x_0,y_0,z_0)
- $f_{xy} = f_{yx}$ Clairaut's theorem, if f_{xy} and f_{yx} are continuous.
- $r_u(u,v)$, $r_v(u,v)$ tangent to surface parameterized by $r(u,v)$

Partial Differential Equations

- $f_t = f_{xx}$ heat equation
- $f_{tt} - f_{xx} = 0$ wave equation
- $f_x - f_t = 0$ transport equation
- $f_{xx} + f_{yy} = 0$ Laplace equation
- $f_t + ff_x = f_{xx}$ Burgers equation

Gradient

- $\nabla f(x,y) = [f_x, f_y]^T$, $\nabla f(x,y,z) = [f_x, f_y, f_z]^T$, gradient
- $D_v f = \nabla f \cdot v$ directional derivative
- $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r\,'(t)$ chain rule
- $\nabla f(x_0,y_0)$ is orthogonal to the level curve $f(x,y) = c$ containing (x_0,y_0)
- $\nabla f(x_0,y_0,z_0)$ is orthogonal to the level surface $f(x,y,z) = c$ containing (x_0,y_0,z_0)
- $\frac{d}{dt} f(x+tv) = D_v f$ by chain rule
- $(x-x_0)f_x(x_0,y_0,z_0) + (y-y_0)f_y(x_0,y_0,z_0) + (z-z_0)f_z(x_0,y_0,z_0) = 0$ tangent plane
- $f(x,y)$ increases in the $\nabla f/|\nabla f|$ direction. Functions dance upwards.
- $f(x,y,z) = c$ defines $z = g(x,y)$, and $g_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$ implicit diff

Extrema

- $\nabla f(x,y) = [0,0]^T$, critical point or stationary point
- $D = f_{xx} f_{yy} - f_{xy}^2$ discriminant, useful in second derivative test
- $f(x_0,y_0) \geq f(x,y)$ in a neighborhood of (x_0,y_0) local maximum
- $f(x_0,y_0) \leq f(x,y)$ in a neighborhood of (x_0,y_0) local minimum

- $\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = c$, or $\nabla g = 0$ Lagrange equations
- second derivative test: $\nabla f = (0,0), D > 0, f_{xx} < 0$ **local max**, $\nabla f = (0,0), D > 0, f_{xx} > 0$ **local min**, $\nabla f = (0,0), D < 0$ **saddle point**
- $f(x_0, y_0) \geq f(x,y)$ everywhere, global maximum
- $f(x_0, y_0) \leq f(x,y)$ everywhere, global minimum

Double Integrals

- $\int\int_R f(x,y)\, dydx$ double integral
- $\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx$ bottom-to-top region
- $\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dxdy$ left-to-right region
- $\int\int_R f(r,\theta) \boxed{r}\, drd\theta$ polar coordinates
- $\int\int_R |r_u \times r_v|\, dudv$ surface area
- $\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy$ Fubini
- $\int\int_R \boxed{1}\, dxdy$ area of region R
- $\int\int_R f(x,y)\, dxdy$ signed volume of solid bound by graph of f and xy-plane

Triple Integrals

- $\int\int\int_R f(x,y,z)\, dzdydx$ triple integral
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx$ integral over rectangular box
- $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y)\, dzdydx$ type I region
- $\int\int\int_R f(r,\theta,z) \boxed{r}\ dzdrd\theta$ integral in cylindrical coordinates
- $\int\int\int_R f(\rho,\theta,z) \boxed{\rho^2 \sin(\phi)}\ dzdrd\theta$ integral in spherical coordinates
- $\int_a^b \int_c^d \int_u^v f(x,y,z)\, dzdydx = \int_u^v \int_c^d \int_a^b f(x,y,z)\, dxdydz$ Fubini
- $V = \int\int\int_E \boxed{1}\, dzdydx$ volume of solid E
- $M = \int\int\int_E f(x,y,z)\, dzdydz$ mass of solid E with density f.

General advise

- Draw the region when integrating in in higher dimensions.
- Consider other coordinate systems if the integral does not work.
- Consider changing the order of integration if the integral does not work.
- For tangent planes, compute the gradient $[a,b,c]^T$ first then fix the constant.
- When looking at relief problems, mind the gradient.

Theorems

- $f_{xy} = f_{yx}$, Taylor, $\int f dxdy = \int f dydx$, Morse theorem, chain rule, gradient theorem, change of variables

People

- Clairaut, Fubini, Lagrange, Fermat, Riemann, Archimedes, Hamilton, Euler, Taylor, Morse, Hopf