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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Total :

Unit 13: Hourly 1 (III practice)

Problems

Problem 13R.1 (10 points):
Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for every positive integer n.

Solution:
(i) This is true for $n = 1$. Both sides are then 1. It is the induction assumption.
(ii) Assume the formula holds for n, we have to to show it for $n+1$. $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2$ is using the induction assumption $n(n+1)(2n+1))/6 + (n+1)^2$ and this is $(n+1)[n(2n+1)/6 + (n+1)] = (n+2)(2n+3)/6$ as required.

Problem 13R.2 (10 points):

a) (6 points) Row reduce $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

b) (4 points) Find A^2.

Solution:
a) Subtract twice the first row from the third and the sum of the first and second from the last: $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) $\begin{bmatrix} 3 & 0 & 3 & 0 \\ 1 & 2 & 1 & 2 \\ 6 & 0 & 6 & 0 \\ 4 & 2 & 4 & 2 \end{bmatrix}$.

Problem 13R.3 (10 points):
a) (2 points) Parametrize the curve $y(x^2+1) = 1$.
b) (2 points) Parametrize the spiral $r = \theta^2$.
c) (2 points) Parametrize the graph $x = e^y = \exp(y)$
d) (2 points) Parametrize the y-axis $x = 0$.
e) (2 points) Parametrize the ellipse $(x-1)^2/9 + (y-4)^2/4 = 1$.

Solution:
a) $r(t) = \begin{bmatrix} t \\ 1/(1+t^2) \end{bmatrix}$
b) $r(t) = \begin{bmatrix} t^2\cos(t) \\ t^2\sin(t) \end{bmatrix}$.
c) $r(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}$
d) $r(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
e) $r(t) = \begin{bmatrix} 3\cos(t)+1 \\ 2\sin(t)+4 \end{bmatrix}$.

Problem 13R.4 (10 points):
Find the arc length of the curve

$$r(t) = [\sqrt{2}\log(t), \log(t)^2/2, \log(\log(t))]^T$$

for $e \leq t \leq e^2$.

Solution:
$r'(t) = [\sqrt{2}/t, \log(t)/t, 1/(t\log(t))]$. It has length $(\log(t)/t + 1/(t\log(t))$.
The anti derivative is $\log(t)^2/2 + \log(\log(t))$. The integral is $(2 + \log(2)) - 1/2 = (3/2) + \log(2)$.

Problem 13R.5 (10 points):
a) (2 points) If v and w are two non-zero vectors, what is $|v \times w|/|v \cdot w|$?
b) (2 points) Finish this: if f is uniformly continuous on $[a, b]$, if

$$|x - y| \leq \dots, \text{ then } |f(x) - f(y)| \leq M_n.$$

c) (2 points) If $x \cdot Bx + Ax - b = 0$ is a quadratic manifold.
What object is B, What object is A? Be specific. In case of vectors, distinguish between row and column vectors for example.
d) (2 points) Who found the formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$?
e) (2 points) Is it true that if f is a function on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$, then $f'(x) = 0$ at some point?

Solution:
a) $\tan(\alpha)$
b) $1/n$
c) B is a symmetric matrix. A is a row vector.
d) Euler
e) no)

Problem 13R.6 (10 points):
a) (2 points) What is $(3 + i)^3$?
b) (2 points) What is $(-1)^i$?
c) (2 points) What is i^4?
d) (2 points) What surface is in cylindrical coordinates given as $z - r^2 = 1$?
e) (2 points) What set of points satisfies $\phi = \pi$?

Solution:
a) $(3+i)(8+6i) = 18+26i$.
b) $e^{-\pi}$
c) 1
d) A paraboloid.
e) The negative z axes.

Problem 13R.7 (10 points):

a) (5 points) You are given $r''(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ t \end{bmatrix}$ and $r(0) = (0,0,0)$ and $r'(0) = (0,0,0)$. Find $r(2\pi)$.
b) (5 points) What is the curvature of $r(t) = [2\sin(t), 3\cos(t), 0]$ at $t = 0$?

Solution:
a) integrate twice $[t - \sin(t), 1 - \cos(t), t^3/6]$. Evaluate at $t = 2\pi$.
b) $r'(0) = [2,0,0], r''(0) = [0,3,0]$ the curvature is $6/8 = 3/4$.

Problem 13R.8 (10 points):
a) (5 points) Find the parametrization of the surface $x^2 - y^2 + z^2 = 1$.
b) (5 points) Find the parametrization of the surface $x^2/4 + y^2/9 = 1$.

Problem 13R.9 (10 points):
a) (5 points) Find the dot product between the two matrices

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} .$$

b) (5 points) Find the angle between these two matrices.

Solution:
a) $A \cdot B = 3 * 2 + 4 * 1 + 1 * 1 = 11$
b) $A \cdot B/(|A||B|) = 11/(\sqrt{27}\sqrt{6})$

Problem 13R.10 (10 points):
a) (5 points) What is the Jacobian matrix of the coordinate change

$$f(x, y, z) = \begin{bmatrix} x^2 \\ y^2x + yx^2 \\ z^3 \end{bmatrix}$$

.
b) (5 points) What is the distortion factor of f?

Solution:
a) $\begin{bmatrix} 2x & 0 & 0 \\ y^2 & 2xy & 0 \\ 0 & 0 & 3z^2 \end{bmatrix}$.
b) $12x^2yz^2$.