LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 8: Arc length

Lecture

8.1. We assume in this lecture that curves are **continuously differentiable** meaning that the velocity is continuous. We would write $r \in C^1([a,b],\mathbb{R}^d)$. Given a parametrized curve $r(t)$ defined over an interval $I = [a,b]$, its **arc length** is defined as

$$L = \int_a^b |r'(t)| \; dt \; .$$

For $f(t) = |r'(t)|$ the integral is defined as the **lim sup** (we don't know yet whether lim exists),

$$\int_a^b f(t) \; dt = \limsup_{n\to\infty} \frac{S_n}{n} = \limsup_{n\to\infty} \frac{1}{n} \sum_{a \leq \frac{k}{n} < b} f(\frac{k}{n}) \; .$$

This **Archimedes integral** is a special **Riemann integral**. It satisfies $\min(f) \leq (b-a)^{-1} \int_a^b f(t) \; dt \leq \max(f)$. The **intermediate value theorem** implies that there is $y \in [a,b]$ such that $f(y) = (b-a)^{-1} \int_a^b f(t)dt$. The minimum and maximum exists by **Bolzano's extreme value theorem**. Related to Bolzano is the **Heine-Cantor theorem** assuring that a continuous function f on a closed finite interval $[a,b]$ is **uniformly continuous**: there exists a function $M(t)$ satisfying $\lim_{t\to 0} M(t) = 0$ with $|f(x) - f(y)| \leq M(|x-y|)$ for all $x, y \in [a,b]$. Stronger is **Lipschitz continuity**, which is $M(t) = M \cdot t$ for some constant M. The next proof shows in general that continuous functions are Riemann integrable; the limsup is actually a limit:

Theorem: Arc length exists and is independent of the parameterization.

Proof. (i) To see parameter independence, assume a time change $\phi(t)$ with a monotone smooth function $\phi : [a,b] \to [\phi(a), \phi(b)]$. If $r(t)$ on $[\phi(a), \phi(b)]$ and $R(t) = r(\phi(t))$ on $[a,b]$ are the two parametrizations and $f(t) = |r'(t)|$ and $F(t) = |R'(t)| = |r'(\phi(t))|\phi'(t)$, then by substitution, the arc length of $r(t)$ is $\int_{\phi(a)}^{\phi(b)} f(t) \; dt = \int_a^b f(\phi(t))\phi'(t) \; dt$ which is $\int_a^b F(t) \; dt$, the arc length of $R(t)$.
(ii) From (i) we can assume $[a,b] = [0,1]$. By uniform continuity, there are $M_n \to 0$ such that if $|y - x| \leq 1/n$, then $|f(y) - f(x)| \leq M_n$. The **intermediate value theorem**, gives for every $I_k = [x_k, x_{k+1}] = [k/n, (k+1)/n] \subset [0,1]$, a $y_k \in I_k$ such that $\int_{x_k}^{x_{k+1}} f(x) \; dx = f(y_k)/n$. Now, $\int_0^1 f(x) \; dx = (1/n)\sum_k f(y_k)$ and $|S_n/n - \int_0^1 f(x) \; dx| = (1/n)|\sum_k [f(x_k) - f(y_k)]| \leq (1/n)\sum_k |f(x_k) - f(y_k)| \leq 1/n \sum_k M_n = M_n \to 0$. □

Examples

8.2. The arc length of the circle $r(t) = [R\cos(t), R\sin(t)]$ with $t \in [0, 2\pi]$ is $\int_0^{2\pi} |r'(t)|\, dt = \int_0^{2\pi} R\, dt = 2\pi R$.

8.3. The arc length of the parabola $r(t) = [t, t^2/2]$ with $t \in [-1, 1]$ is $\int_{-1}^{1} \sqrt{1+t^2}\, dt$. We will do this integral in class. The result is $\sqrt{2} + \text{arcsinh}(1)$.

8.4. The arc length of the curve $r(t) = [\log(t), \sqrt{2}t, t^2/2]$ for $t \in [1, 2]$. It is $\int_1^2 \sqrt{1/t^2 + t^2 + 2}\, dt = \int_1^2 (t + 1/t)\, dt = \log(2) + 3/2$.

Illustrations

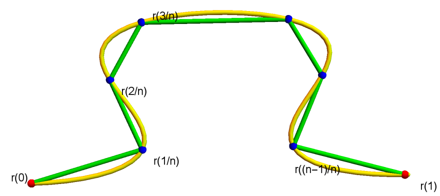

Figure 1. A polygon approximation of a curve produces a Riemann sum approximation of the length integral.

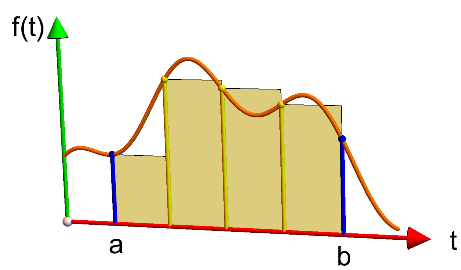

Figure 2. A Riemann sum approximation of a continuous function produces in the limit the “area under the curve”.

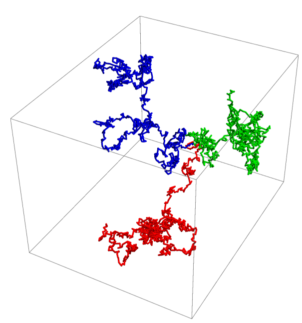

Figure 3. Brownian motion produces continuous paths which are not differentiable. The arc length integral does not exist.

Background (cool, but can be ignored if you like)

8.5. A function f is called **Lipschitz continuous** on $[a,b]$ if there exists a constant M such that $|f(t)-f(s)| \leq M|t-s|$ for all $t,s \in [a,b]$. It turns out that for Lipschitz functions the derivative f' exists "almost everywhere". To make sense of this, a more mature integration theory is needed. The Riemann integral is totally inadequate and does not fit the bill. We need the **Lebesgue integral**. The fundamental theorem of calculus for Lipschitz function is known as the **Rademacher theorem**:

> **Theorem:** If f is Lipschitz, then $\int_a^b f'(t)\,dt = f(b)-f(a)$.

8.6. In order to define the **Lebesgue integral**, one first introduces a so called **σ-algebra $\mathcal{A}$**. [1] It is the smallest set of subsets of $\mathbb{R}$ which is closed under the operation of taking **countable unions and intersections** and complements and which contains the class of intervals. The Lebesgue measure on intervals $|[a,b]| = b-a$ can then be extended to $\mathcal{A}$ where it inherits all the properties we want, like $|A \cup B| = |A| + |B| - |A \cap B|$. For **indicator functions** which are functions $f(x) = 1_A(x)$ which is 1 if $x \in A$ and 0 else, the Lebesgue integral is defined as $\int 1_A(x)\,dx = |A|$.

8.7. First write the function f as $f^+ - f^-$, where f^+ and f^- are both non-negative. This is a simplification because we need to define the integral only for non-negative functions. A **simple step function** is a finite sum $\sum_i a_i 1_{A_i}$, with $A_i \in \mathcal{A}$. For such functions, define $\int_I f dx = \sum_i a_i |A_i|$. The **Lebesgue integral** is now defined as $\sup_{g \leq f} \int_I g\,dx$, where the supremum is taken over all simple step functions g smaller or equal than f. If the limit exists, the function is called **Lebesgue integrable**.

8.8. The Lebesgue integral is also a **Monte Carlo integral** $\lim_{n\to\infty} \frac{1}{n} \sum_{a \leq x_k < b} f(x_k)$, where x_k are random choices in $[a,b]$. This is justified by the **law of large numbers.** The transition Riemann $\to$ Lebesgue replaces a regular lattice k/n with a random one.

8.9. The Lebesgue integral can integrate also non-continuous functions: let $g(x)$ be 0 on rational numbers and 1 on irrational numbers. Then $\int_I g\,dx = |I|$ because all except a countable number of x are irrational. The Riemann integral would give 0.

8.10. The proof that a continuous function is Lebesgue integrable is even simpler than for the Riemann integral: first again use that f is **uniform continuous** on $[a,b]$, there exists $M_n \to 0$ such that whenever $|x-y| \leq 1/n$, also $|f(x)-f(y)| \leq M_n$. Take the intervals $I_k = [k/n,(k+1)/n] \cap [a,b]$ and step functions $g = \sum_k c_k 1_{I_k}$ and $h = \sum_k d_k 1_{I_k}$, where c_k is the minimum of f on I_k and d_k the maximum. Now $\int_a^b |g-h|\,dx \leq \sum_k |c_k - d_k||I_k| \leq M_n \sum_k |I_k| = M_n(b-a)$. Now f is sandwiched between step functions g,h which for $n\to\infty$ have the same integral.

8.11. We don't prove Rademacher here. One needs to show that f' is Lebesgue integrable and that $g(x) = f(a) + \int_a^x f'(t)\,dt$ agrees with $f(x)$. In modern language Rademacher tells **Lipschitz** = **Sobolev** $W^{1,\infty}([a,b]) = \{f' \in L^\infty([a,b])\}$. More general is **absolute continuity** = $W^{1,1}([a,b]) = \{f' \in L^1([a,b])\}$.

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[1] For details see i.e. O.Knill, Probability theory and stochastic processes, 2011

Homework

Problem 8.1: Find the arc length of the **catenary** $r(t) = [t, \cosh(t)]$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. Hint. You can use the identity $\cosh^2(t) - \sinh^2(t) = 1$, where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. We have $\cosh' = \sinh, \sinh' = \cosh$.

Galileo was the first to investigate the catenary. It is the curve, a freely hanging heavy rope describes, if the end points have the same height. Galileo mistook the curve for a parabola. It was Johannes Bernoulli in 1691, who obtained its true form after some competition involving Huygens, Leibniz and two Bernoullis. The name "catenarian" (=chain curve) was first used by Huygens in a letter to Leibnitz in 1690.

Problem 8.2: Find the arc length of the cycloid

$$r(t) = [t - \sin(t), 1 + \cos(t)]$$

from 0 to 2π. The upside down cycloid is the solution to the famous **Brachistochrone problem**, the curve along which a ball descends fastest. Hint. You might want to use the double angle formula $2 - 2\cos(t) = 4\sin^2(\frac{t}{2})$.

Problem 8.3: Find the length of the curve

$$r(t) = [12t, 8t^{3/2}, 3t^2] ,$$

where $t \in [0, 3]$.

Problem 8.4: Compute numerically the arc length of the knot $r(t) = [\sin(4t), \sin(3t), \cos(5t), \cos(7t)]$ from $t = 0$ to $t = 2\pi$. By drawing the first coordinates only and using color as the fourth coordinate, we can see that there are no non-trivial knots in $\mathbb{R}^4$. You can not tie your shoes in $\mathbb{R}^4$!

Problem 8.5: What is the relation between $|\int_0^1 r'(t) \quad dt|$ and $\int_0^1 |r'(t)| \; dt$? Give an interpretation of both sides.

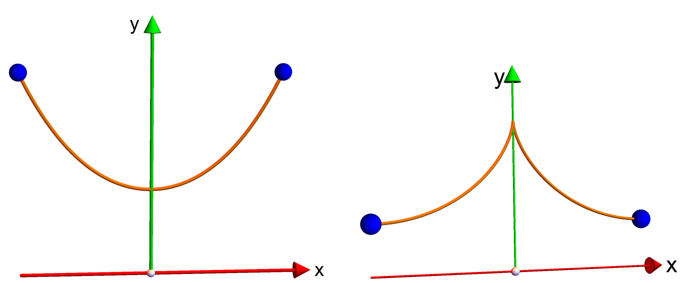

Figure 4. The catenary and the cycloid.