LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 6: Visual proofs

Seminar

6.1. Geometric intuition and pictures allow to prove results visually. An example:

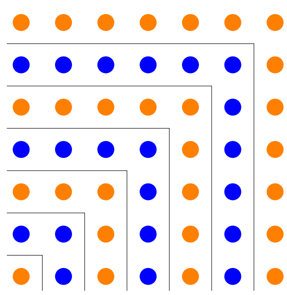

FIGURE 1. This is a proof without words.

[1]

Problem A: What formula does Figure (1) prove?

6.2. By drawing a rectangle of side length a and b, we can see that the area $a * b$ is the same as the area $b * a$. For the cross product or matrices, this is wrong.

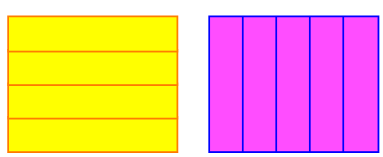

FIGURE 2. A Cuisenaire proof that $4 * 5 = 5 * 4$. Four yellow sticks of length 5 have the same area than 5 purple sticks of length 4.

[1] Cover of the book "Proofs without words"

6.3. Pictures help to get intuition about a mathematical result. The Pythagorean theorem was first proven geometrically. The visual proof we look at here could well have been the first which was found.

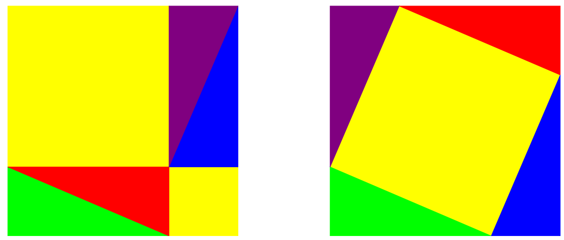

FIGURE 3. A visual proof of the Pythagorean theorem. It is probably one of the first proofs.

Problem B: Use Figure (3) for a proof of the Pythagorean theorem. You can either describe in words, or label some parts of the picture. Remember that we want to show $c^2 = a^2 + b^2$.

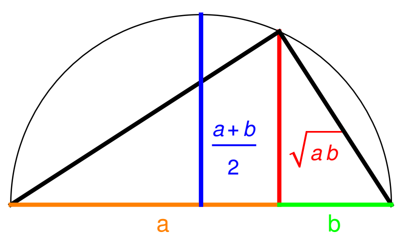

FIGURE 4. A visual proof of $\sqrt{ab} \leq (a+b)/2$.

6.4. The **geometric-algebraic inequality** assures that the geometric mean is smaller or equal than the algebraic mean. In order to appreciate that proof, we have first to verify an identity relating the lengths a, b cut by the altitude line and height h.

Problem C: First check why the triangle in Figure 4 is a right angle. Then use Pythagoras three times to prove $ab = h^2$. Finally check the geometric-algebraic inequality.

[2]C. Gallant, Mathematics Magazine, 50(2), 1977, page 98

6.5.

Theorem: The radius of the inscribed circle in a 3 : 4 : 5 triangle is 1

Problem D: Use Figure (5) from the "9 Chapters" to prove the theorem.

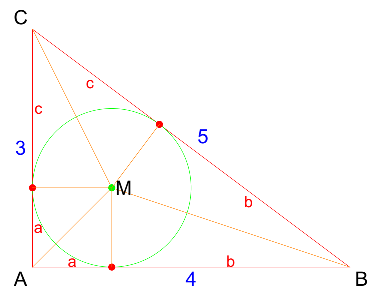

FIGURE 5. The 3-4-5 triangle. Can you use the picture to prove that a=1?

6.6. Find the formula for the volume of a tetrahedron given by 4 points A, B, C, D.

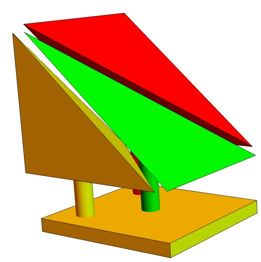

FIGURE 6. The tetrahedron volume is 1/6 of a parallelepiped volume. Not only the Egyptions knew it, this figure can also be found in the "nine chapters". We build a statue which can be 3D printed.

[3]

Problem E: Use Figure (6) to prove that the volume is a sixth of the volume of the corresponding parallelepiped.

[3]"Illustrating Mathematics using 3D printers", by O. Knill and E. Slavkovsky.

Homework

Exercices A-D are done in the seminar. This homework is due on Tuesday:

Problem 6.1 The **3D Pythagoras theorem** states that the square of the area of ABC is the sum of the squares of the areas of the triangles OAB, OBC and OCA (which are each half of a rectangle). Use Figure (7) with $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$ to verify this theorem. Use the cross product to get the areas.

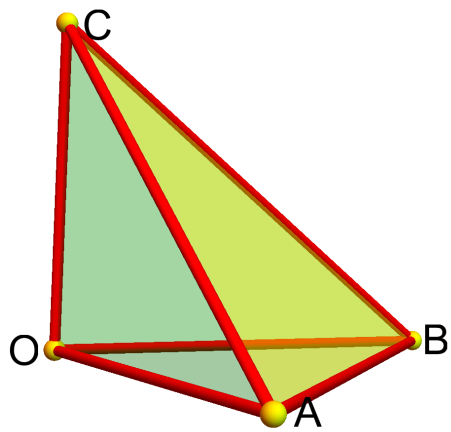

Figure 7. The 3D Pythagoras theorem.

Problem 6.2 Find a formula for the distance of a point P to a line through two points A, B. The final formula should not use any trig functions.

Problem 6.3 Find a formula for the distance of a point P to a plane through three points A, B, C. The final formula should not use any trig functions.

Problem 6.4 Find a formula for the distance between the line through a point A, B and a line through the point C, D. The final formula should not use any trig functions.

Problem 6.5 Look up the rules for quaternion multiplication $(u_0, u_1, u_2, u_3) \star (v_0, v_1, v_2, v_3)$ and verify that $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-v \cdot w, v \times w)$. Historically, this is an important identity as the dot and cross product have been introduced together in the form of quaternions.