The set of piecewise smooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space X. There is an **inner product** in X defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$$

It allows to define angles, length, distance, projections in X as we did in finite dimensions.

THE FOURIER BASIS.

THEOREM. The functions $\{\cos(nx), \sin(nx), 1/\sqrt{2}\,\}$ form an orthonormal set in X.

Proof. You check the details in the homework. To check linear independence a few integrals need to be computed. For all $n, m \geq 1$, with $n \neq m$ you have to show:

$$\langle 1/\sqrt{2}, 1/\sqrt{2}\,\rangle = 1$$
$$\langle \cos(nx), \cos(nx) \rangle = 1, \langle \cos(nx), \cos(mx) \rangle = 0$$
$$\langle \sin(nx), \sin(nx) \rangle = 1, \langle \sin(nx), \sin(mx) \rangle = 0$$
$$\langle \sin(nx), \cos(mx) \rangle = 0$$
$$\langle \sin(nx), 1/\sqrt{2} \rangle = 0$$
$$\langle \cos(nx), 1/\sqrt{2} \rangle = 0$$

To verify the above integrals, the following trigonometric identities are useful:

$$\begin{aligned} 2\cos(nx)\cos(my) &= \cos(nx - my) + \cos(nx + my) \\ 2\sin(nx)\sin(my) &= \cos(nx - my) - \cos(nx + my) \\ 2\sin(nx)\cos(my) &= \sin(nx + my) + \sin(nx - my) \end{aligned}$$

FOURIER COEFFICIENTS. The **Fourier coefficients** of a function f in X are defined as

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2}\, dx$$

$$a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

$$b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$$

FOURIER SERIES. The **Fourier representation** of a piecewise smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

We take it for granted that the series converges and that the identity holds at all points x where f is continuous.

ODD AND EVEN FUNCTIONS. The following advice can save you time when computing Fourier series:

If f is odd: $f(x) = -f(-x)$, then f has a sin series.
If f is even: $f(x) = f(-x)$, then f has a cos series.
If you integrate an odd function over $[-\pi, \pi]$ you get 0.
The product of two odd functions is even, the product between an even and an odd function is odd.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function ($f(-x)+f(x) = 0$) so that it has a sin series: with $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{-1}{\pi}(x\cos(nx)/n + \sin(nx)/n^2|_{-\pi}^{\pi}) = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n} \sin(nx)$. If we evaluate both sides at a point x, we obtain identities. For $x = \pi/2$ for example, we get

$$\frac{\pi}{2} = 2(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots)$$

is a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series. There is no need to compute the integrals. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

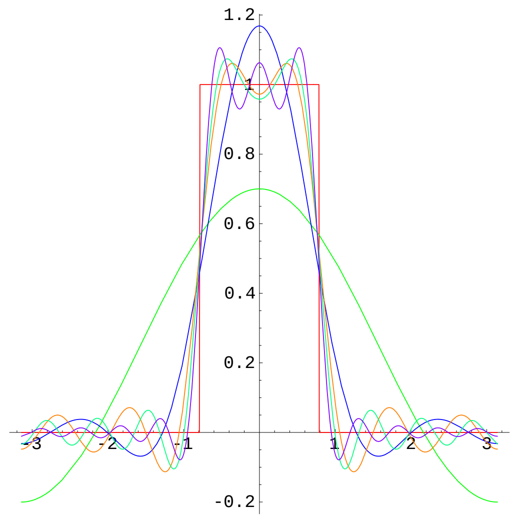
EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos(nx)\,dx = \frac{\sin(nx)}{\pi n}|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m+1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - ...)$$

Remark. The function in Example 3 is not smooth, but Fourier theory still works. What happens at the discontinuity point $\pi/2$? The Fourier series converges to $1/2$. Diplomatically it has chosen the point in the middle of the limits from the right and the limit from the left.

FOURIER APPROXIMATION. For a smooth function f, the Fourier series of f converges to f. The Fourier coefficients are the coordinates of f in the Fourier basis.

The function $f_n(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^n a_k\cos(kx) + \sum_{k=1}^n b_k\sin(kx)$ is called a **Fourier approximation** of f. The picture to the right plots a few approximations in the case of a piecewise continuous even function given in example 3).



THE PARSEVAL EQUALITY. When evaluating the square of the length of f with the square of the length of the series, we get

$$||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2 .$$

EXAMPLE. We have seen in example 1 that $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ Because the Fourier coefficients are $b_k = 2(-1)^{k+1}/k$, we have $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = 2\pi^2/3$ and so

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ... = \frac{\pi^2}{6}$$

Isn't it fantastic that we can sum up the reciprocal squares? This formula has been obtained already by **Leonard Euler**. The problem was called the **Basel problem**.

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** f in time t. While the governing equation for a vector was an ordinary differential equation $\dot{x} = Ax$ (ODE), the describing equation is now be a **partial differential equation** (PDE) $\dot{f} = T(f)$. The function $f(x,t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

Example: for $f(x,t) = \cos(x + 4t^2)$, we have
- $f_x(x,t) = -\sin(x + 4t^2)$
- $f_t(x,t) = -8t\sin(x + 4t^2)$.
- $f_{xx}(x,t) = -\cos(x + 4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,t)$ for the partial derivative with respect to x and $f_t(x,t)$ for the partial derivative with respect to t.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which different partial derivatives occur.

- $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x - t)$.
- $f_{tt}(x,t) - f_{xx}(x,t) = 0$ with $f(x,0) = \sin(x)$ and $f_t(x,0) = 0$ has a solution $f(x,t) = (\sin(x - t) + \sin(x + t))/2$.

THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal bar $[0,\pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x,t)$ at x. The function $f(x,t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x,t)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ.

REWRITING THE PROBLEM. We can write the problem as

$$\boxed{\frac{d}{dt} f = \mu D^2 f}$$

We will solve the problem in the same way as we solved linear differential equations:

$$\boxed{\frac{d}{dt}\vec{x} = A\vec{x}}$$

where A is a matrix - $\boxed{\text{by diagonalization}}$.

We use that the Fourier basis is just the diagonalization: $D^2\cos(nx) = (-n^2)\cos(nx)$ and $D^2\sin(nx) = (-n^2)\sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue $\boxed{-n^2}$. By a symmetry trick, we can focus on sin-series from now on.

SOLVING THE HEAT EQUATION WITH FOURIER THEORY. The heat equation $f_t(x,t) = \mu f_{xx}(x,t)$ with smooth $f(x,0) = f(x)$, $f(0,t) = f(\pi,t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$$

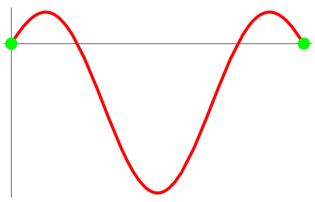
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

Proof: With the initial condition $f(x) = \sin(nx)$, we have the evolution $f(x,t) = e^{-\mu n^2 t} \sin(nx)$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ then $f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$.

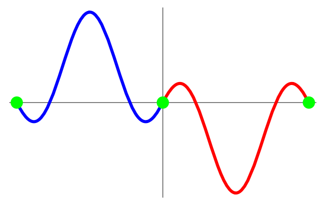
A SYMMETRY TRICK. Given a function f on the interval $[0,\pi]$ which is zero at 0 and π. It can be extended to an odd function on the doubled integral $[-\pi,\pi]$.

The Fourier series of an odd function is a pure sin-series. The Fourier coefficients are $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$.

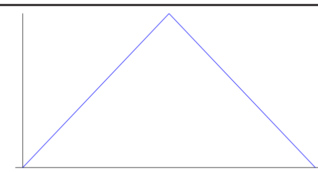
The function is given on $[0,\pi]$.



The odd symmetric extension on $[-\pi,\pi]$.



EXAMPLE. Assume the initial temperature distribution $f(x,0)$ is a sawtooth function which has slope 1 on the interval $[0,\pi/2]$ and slope -1 on the interval $[\pi/2,\pi]$. We first compute the sin-Fourier coefficients of this function.

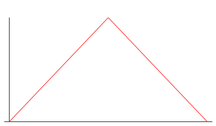


The sin-Fourier coefficients are $b_n = \frac{4}{n^2\pi}(-1)^{(n-1)/2}$ for odd n and 0 for even n. The solution is

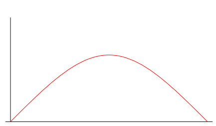
$$f(x,t) = \sum_{n}^{\infty} b_n e^{-\mu n^2 t} \sin(nx) \; .$$

The exponential term containing the time makes the function $f(x,t)$ converge to 0: The body cools. The higher frequencies are damped faster: "smaller disturbances are smoothed out faster."

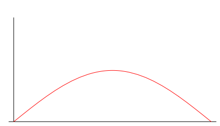
VISUALIZATION. We can plot the graph of the function $f(x,t)$ or slice this graph and plot the temperature distribution for different values of the time t.



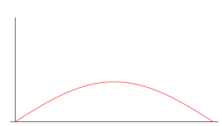
$f(x,0)$



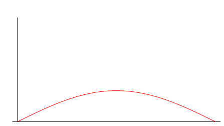
$f(x,1)$



$f(x,2)$



$f(x,3)$



$f(x,4)$

THE WAVE EQUATION. The height of a string $f(x,t)$ at time t and position x on $[0,\pi]$ satisfies the **wave equation**

$$f_{tt}(x,t) = c^2 f_{xx}(x,t)$$

where c is a constant. As we will see, c is the **speed** of the waves.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d^2}{dt^2} f = c^2 D^2 f$$

We will solve the problem in the same way as we solved

$$\frac{d^2}{dt^2} \vec{x} = A\vec{x}$$

If A is diagonal, then every basis vector x satisfies an equation of the form $\frac{d^2}{dt^2} x = -c^2 x$ which has the solution $x(t) = x(0)\cos(ct) + x'(0)\sin(ct)/c$.

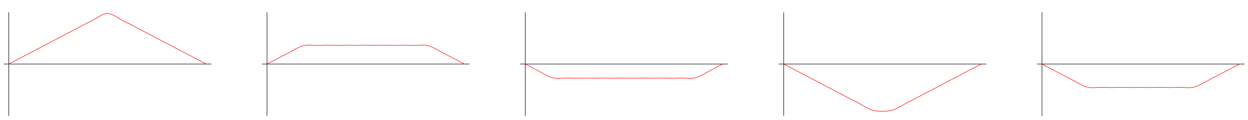
SOLVING THE WAVE EQUATION WITH FOURIER THEORY. The wave equation $f_{tt} = c^2 f_{xx}$ with $f(x,0) = f(x)$, $f_t(x,0) = g(x)$, $f(0,t) = f(\pi,t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\, dx$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\, dx$$

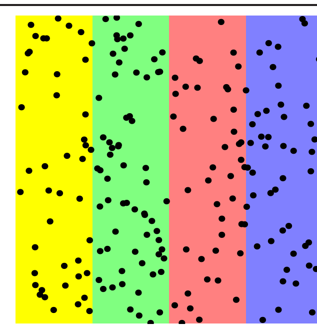
Proof: With $f(x) = \sin(nx), g(x) = 0$, the solution is $f(x,t) = \cos(nct)\sin(nx)$. With $f(x) = 0, g(x) = \sin(nx)$, the solution is $f(x,t) = \frac{1}{nc}\sin(cnt)\sin(nx)$. For $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ and $g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, we get the formula by summing these two solutions.

VISUALIZATION. We can just plot the graph of the function $f(x,t)$ or plot the string for different times t.

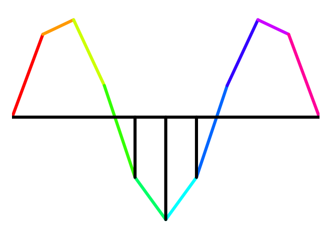


$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x,t)$ is proportional to the kinetic energy at x. Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly either to the right or to the left. If $f_i(t)$ is the **energy** of particles in cell i at time t, then the energy $f_i(t+dt)$ of particles at time $t+dt$ is proportional to $f_i(t) + (f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$. Therefore the **discrete time derivative** $f_i(t+dt) - f_i(t) \sim dt f_t$ is equal to the **discrete second space derivative** $dx^2 f_{xx}(x,t) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.



TO THE DERIVATION OF THE WAVE EQUATION. We can model a string by n discrete particles linked by springs. Assume that the particles can move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i+1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$ which is a discrete version of the second derivative because $f_{xx}(x,t)dx^2 \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.



OVERVIEW: The heat and wave equation can be solved like ordinary differential equations:

Ordinary differential equations	Partial differential equations
$x_t(t) = Ax(t)$ $x_{tt}(t) = Ax(t)$	$f_t(x,t) = f_{xx}(x,t)$ $f_{tt}(x,t) = f_{xx}(x,t)$
Diagonalizing A leads for eigenvectors $\vec{v}$ $Av = -c^2 v$ to the differential equations $v_t = -c^2 v$ $v_{tt} = -c^2 v$ which are solved by $v(t) = e^{-c^2 t} v(0)$ $v(t) = v(0)\cos(ct) + v_t(0)\sin(ct)/c$	Diagonalizing $T = D^2$ with eigenfunctions $f(x) = \sin(nx)$ $Tf = -n^2 f$ leads to the differential equations $f_t(x,t) = -n^2 f(x,t)$ $f_{tt}(x,t) = -n^2 f(x,t)$ which are solved by $f(x,t) = f(x,0)e^{-n^2 t}$ $f(x,t) = f(x,0)\cos(nt) + f_t(x,0)\sin(nt)/n$

NOTATION:

f function on $[-\pi,\pi]$ smooth or piecewise smooth.
t time variable
x space variable
D the differential operator $Df(x) = f'(x) = d/dx f(x)$.
T linear transformation, like $Tf = D^2 f = f''$.
c speed of the wave.

$Tf = \lambda f$ Eigenvalue equation analog to $Av = \lambda v$.
f_t partial derivative of $f(x,t)$ with respect to time t.
f_x partial derivative of $f(x,t)$ with respect to space x.
f_{xx} second partial derivative of f twice with respect to space x.
μ heat conductivity
$f(x) = -f(-x)$ odd function, has sin Fourier series