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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F**

If A is a non-invertible $n \times n$ matrix, then $\det(A) \neq \det(\text{rref}(A))$.

Solution:
A as well as $\text{rref}(A)$ have determinant 0.

2) **T** F

If the rows of a square matrix form an orthonormal basis, then the columns must also form an orthonormal basis.

Solution:
You have proven that $AA^T = 1_n$ is equivalent to $A^T A = 1_n$.

3) **T** F

A 3×3 matrix A for which the sum of the first two columns is the third column has zero determinant.

Solution:
The matrix is not invertible in that case.

4) T **F**

A 2×2 rotation matrix $A \neq I_2$ does not have any real eigenvalues.

Solution:
The rotation by an angle π has the real eigenvalue -1 with algebraic multiplicity 2.

5) **T** F

If A and B both have $\vec{v}$ as an eigenvector, then $\vec{v}$ is an eigenvector of AB.

Solution:
Check $AB\vec{v} = \mu\lambda\vec{v}$ if λ is the eigenvalue to $\vec{v}$ of A and μ is the eigenvalue of $\vec{c}$ with respect to B.

6) T **F**

If A and B both have λ as an eigenvalue, then λ is an eigenvalue of AB.

Solution:
Already diagonal matrices can be used for counter examples, like $A = \text{diag}(2, 1/2), B = \text{diag}(1/2, 2)$. It is even not true for nonzero 1×1 matrices.

7) T **F** Similar matrices have the same eigenvectors.

Solution:
They have the same eigenvalues, not eigenvectors

8) **T** F If a 3×3 matrix A has 3 independent eigenvectors, then A is similar to a diagonal matrix.

Solution:
According to one of the basic results

9) **T** F If a square matrix A has non-trivial kernel, then 0 is an eigenvalue of A.

Solution:
$A\vec{v} = 0$ means also $A\vec{v} = 0\vec{v}$.

10) **T** F If the rank of an $n \times n$ matrix A is less than n, then 0 is an eigenvalue of A.

Solution:
The kernel is then not trivial by the dimension formula.

11) **T** F Two diagonalizable matrices whose eigenvalues are equal must be similar.

Solution:
Both matrices are similar to a common diagonal matrix.

12) T **F** A square matrix A is diagonalizable if and only is A^2 is diagonalizable.

Solution:
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counter example. It is not diagonalizable but $A^2 = 0$ is.

13) **T** F If a square matrix A is diagonalizable, then $(A^T)^2$ is diagonalizable.

Solution:
$S^1 AS = B$, then $S^{-1}A^2S = B^2$ and $S^T(A^T)^2(S^{-1})^T = (B^T)^2$ is diagonal.

14) T **F** The matrices $\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ are similar.

Solution:
The geometric multiplicities are different.

15) **T** F There exist matrices A with k distinct eigenvalues whose rank is strictly less than k.

Solution:
Just make sure that one eigenvalue is 0.

16) **T** F If A is an $n \times n$ matrix which satisfies $A^k = 0$ for some positive integer k, then all the eigenvalues of A are 0.

Solution:
If not, we would have a nonzero eigenvalue λ with eigenvector $\vec{v}$ and $A^n\vec{v} = \lambda^n\vec{v}$ is nonzero so that A^n can not be zero.

17) T **F** If a 3×3 matrix A satisfies $A^2 = I_3$ and A is diagonalizable, then A must be similar to the identity matrix.

Solution:
The matrix $A = -I_n$ is diagonalizable and satisfies $A^2 = I_3$, but A has different eigenvalues as the identity matrix and can not be similar to the identity matrix.

18) T **F** A and A^T have the same eigenvectors.

Solution:
They have the same eigenvalues.

19) **T** F The least squares solution of a system $A\vec{x} = \vec{b}$ is unique if and only if $\ker(A) = 0$.

Solution:
If $\ker(A) = 0$, you have seen the formula for the solution. If $\ker(A)$ is not trivial and $\vec{v}$ is in the kernel, then any least square solution $\vec{x}$ has also $\vec{x} + \vec{v}$ as a least square solution.

20) T F The matrix $\begin{bmatrix} 1 & 1000 & 1 & 1 \\ 1000 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1000 \\ 1 & 1 & 1000 & 1 \end{bmatrix}$ is invertible.

Solution:
Do Lagrange expansion.

Problem 2) (10 points)

Match the following matrices with the correct label. No justifications are needed. Fill in a),b),c),d),e) into the boxes.

A) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 9 \end{bmatrix}$ ☐

B) $\begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & 7 \\ -4 & -7 & 0 \end{bmatrix}$ ☐

C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ ☐

D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ☐

E) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ☐

a) (2 points) skewsymmetric matrix
b) (2 points) nondiagonalizable matrix
c) (2 points) orthogonal projection

d) (2 points) symmetric matrix
e) (2 points) orthogonal matrix

Solution:
a) - B)
b) - E)
c) - D)
d) - A)
e) - C)

Problem 3) (10 points)

Find a basis for the subspace V of $\mathbf{R}^4$ given by the equation $x + 2y + 3z + 4w = 0$. Find the matrix which gives the orthogonal projection onto this subspace.

Solution:

1. Solution.

Form $A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ which has the orthogonal complement as the image. The matrix $P' = A(A^TA)^{-1}A^T$ is the projection onto this orthogonal complement. We get $(A^TA)^{-1} = 1/30$ and $P' = AA^T/30 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}/30$ so that $P = 1 - P'$ and

$$P = \begin{bmatrix} 29 & -2 & -3 & -4 \\ -2 & 26 & -6 & -8 \\ -3 & -6 & 21 & -12 \\ -4 & -8 & -12 & 14 \end{bmatrix}/30.$$

2. Solution.

Find directly an orthogonal basis in V, for example by choosing two already orthogonal vectors $[-2, 1, 0, 0], [0, 0, -4, 3]$ and searching for a third $[b, 2b, 3a, 4a]$ which is orthogonal to the first two vectors and in V if $b + 4b + 9a + 16a = 0$, which gives $25a = -5b$ or $5a = b$ and so $[5, 10, -3, -4]$. These vectors can be made orthonormal $[-2, 1, 0, 0]/\sqrt{5}, [0, 0, -4, 3]/5, [5, 10, -3, -4]/(5\sqrt{6})$. We can now form the matrix $A = \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & \sqrt{2/3} \\ 0 & -4/5 & -\sqrt{3/2}/5 \\ 0 & 3/5 & -2\sqrt{2/3}/5 \end{bmatrix}$ and then AA^T and obtain $P = AA^T$.

3. Solution.

Take three arbitrary linearly independent vectors in V, form the matrix A which contains these vectors as columns and then calculate $A(A^TA)^{-1}A^T$. While routine, this gives quite a bit of work, more than in the 1. and 2. Solutions.

Problem 4) (10 points)

Assume that A is a skew-symmetric matrix, that is, it is a $n \times n$ matrix which satisfies $A^T = -A$.

a) Find $\det(A)$ if n is odd.

b) What possible values can $\det(A)$ have if $n = 2$?

c) Verify that if λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

Solution:
a) Because $\det(A^T) = \det(A) = (-1)^n \det(-A)$, the determinant is 0 if n is odd.
b) A sqew symmetric matrix has purely imaginary eigenvalues. Because they occur in pairs, the determinant is nonnegative. For $n = 2$, we have the matrix $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$ which has determinant a^2.
c) We know that λ is also an eigenvalue of A^T (because the characteristic polynomials of A and A^T are the same). If $A^T\vec{v} = \lambda\vec{v}$, then $A\vec{v} = -A^T\vec{v} = -\lambda\vec{v}$ so that $-\lambda$ is also an eigenvalue.

Problem 5) (10 points)

The recursion

$$u_{n+1} = u_n - u_{n-1} + u_{n-2}$$

is equivalent to the discrete dynamical system

$$\begin{bmatrix} u_{n+1} \\ u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \\ u_{n-2} \end{bmatrix}.$$

a) Find the (real or complex) eigenvalues of A.

b) Is there a vector $\vec{v}$ such that $||A^n\vec{v}|| \to \infty$?

c) Can you find any positive integer k such that $A^k = I_3$?

Solution:
a) The eigenvalues are $1, -i, i$. Call the eigenvectors $\vec{a}, \vec{b}$ and $\vec{c}$.
b) No, there is no such vector. If $\vec{v} = a\vec{a} + b\vec{b} + c\vec{c}$ we can write down the explicit solution $A^n\vec{v} = a1^n\vec{a} + b(-i)^n\vec{b} + ci^n\vec{c}$.
c) Yes, the matrix is diagonalizable (it has 3 different eigenvalues) and is therefore similar to a diagonal matrix B which has diagonal entries $1, i, -i$. Because $B^4 = I_3$ we also have $A^4 = I_3$.

Problem 6) (10 points)

Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

a) Find $\det(A)$.

b) Find all eigenvalues whether real or complex of A and state their algebraic multiplicities.

c) For each real eigenvalue λ of A find the eigenspace and the geometric multiplicity.

Solution:
a) Q is orthogonal with determinant is -1, there is only one pattern and the signature of this pattern is (-1) because there are three inversions in the pattern.
b) The characteristic polynomial is $f_A(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i)$.
Each eigenvalue $1, -1, i, -i$ has algebraic multiplicity 1.
c) The vector $[1, 1, 1, 1]^T$ is an eigenvector to the eigenvalue 1, the vector $[1, -1, 1, -1]$ is an eigenvector to the eigenvalue -1. The geometric multiplicity is 1.

Problem 7) (10 points)

Find S and a diagonal matrix B such that $S^{-1}AS = B$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix} .$$

Solution:
The eigenvector to the eigenvalue 1 is $[2, -2, 1]^T$.
The eigenvector to the eigenvalue 2 is $[0, -1, 2]^T$.
The eigenvector to the eigenvalue 3 is $[0, 0, 1]$.
The matrix S has the eigenvectors in the columns. The matrix B has the eigenvalues of A in the diagonal: so

$$S = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} .$$

Problem 8) (10 points)

Find the function of the form

$$f(t) = a\sin(t) + b\cos(t) + c$$

which best fits the data points $(0, 0), (\pi, 1), (\pi/2, 2), (-\pi, 3)$.

Solution:
To fit the function, set up the linear equations, for which we want to find the least square solution:

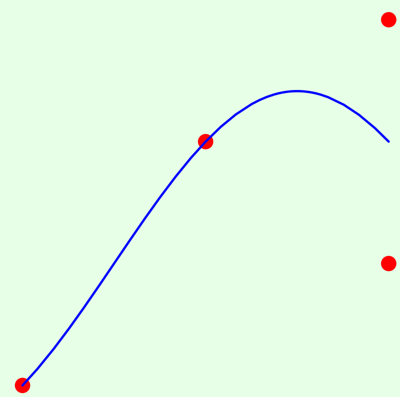
$$\begin{array}{rcl} a\sin(0)+b\cos(0)+c &=& 0 \\ a\sin(\pi)+b\cos(\pi)+c &=& 1 \\ a\sin(\pi/2)+b\cos(\pi/2)+c &=& 2 \\ a\sin(-\pi)+b\cos(-\pi)+c &=& 3 \end{array}$$

which is

$$\begin{array}{rcl} b+c &=& 0 \\ -b+c &=& 1 \\ a+c &=& 2 \\ -b+c &=& 3 \end{array}$$

Solution:
or in matrix form

$$A\vec{x} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b}\,.$$



We get the unknown coefficients

$$\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\,.$$

The solution is $1 - \cos t + \sin(t)$.

Problem 9) (10 points)

Let V be the image of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}\,.$$

a) Find the matrix P of the orthogonal projection onto V.

b) Find the matrix P' of the orthogonal projection on to $V^\perp$.

Solution:
a) Just use the formula $P = A(A^TA)^{-1}A^T$. We get

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

b) The formula is verified by applying P and P' to it. We have $P' = 1_4 - P$. So

$$\begin{bmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \end{bmatrix}.$$

The key to the problem is that the projection P' onto the orthogonal complement V' of a linear space V satisfies $P + P' = I_n$. You can see this by going into an orthonormal basis $\{v_1, ..., v_n\}$ of R^n, where the first k vectors $\{v_1, ..., v_k\}$ form a basis in V and the vectors $\{v_{k+1}, ..., v_n\}$ form a basis in V'. Now $x = (x \cdot v_1)v_1 + ... + (x \cdot v_n)v_n$ and this can be split up into $[(x \cdot v_1)v_1 + ... + (x \cdot v_k)v_k] + [(x \cdot v_{k+1})v_{k+1} + ... + (x \cdot v_n)v_n]$ which is $Px + P'x$.