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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F There is a real diagonalizable 3×3 matrix for which the algebraic multiplicity of an eigenvalue $\lambda = 2$ is larger than 1.

Solution:
You can find a diagonal example.

2) T F Two symmetric not-invertible 3×3 matrices A, B are similar if their trace agrees.

Solution:
They both can be diagonalized.

3) T F Every orthogonal 5×5 matrix has a real eigenvalue.

Solution:
The characteristic polynomial is odd.

4) T F The real eigenvalues of a 4×4 matrix A do not change under row reduction.

Solution:
Already a scaling does change the eigenvalues.

5) T F Every real diagonalizable 3×3 matrix can be diagonalized using an orthogonal matrix S.

Solution:
It is symmetric

6) T F The eigenvalues of a 2×2 rotation matrix are always either 1 or -1.

Solution:
Can be complex

7) T F The nullity of a $n \times n$ matrix A is the same as the nullity of A^T.

Solution:
The nullity of A is the same than the nullity of A^T.

8) T F A discrete dynamical system $\vec{v}(t+1) = A\vec{v}(t)$ defined by a 2×2 matrix A is asymptotically stable if $A^8 = I_2/2$.

Solution:
By definition.

9) T F A 3×3 matrix for a reflection about a line never has trace 0.

Solution:
The diagonal entries add up to -1.

10) T F The trace of a 5×5 orthogonal projection matrix is the dimension of the image.

Solution:
The trace does not change under diagonalization. The trace in diagonal form is the dimension of the image.

11) T F If A and B are diagonalizable $n \times n$ matrices, then AB is diagonalizable.

Solution:
Start with a non-diagonalizable C like $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ and take $B = Diag(1,2)$ for example. Now form $C.B^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

12) T F The sum of the geometric multiplicities of a $n \times n$ matrix A is always at most n.

Solution:
The geometric multiplicities are smaller or equal than the algebraic multiplicities.

13) T F

A 2×2 matrix B such that $\text{tr}(B) = 3$, $\text{tr}(B^2) = 5$ is diagonalizable.

Solution:
The two eigenvalues are different

14) T F

There is a 3×3 matrix A such that $A + 2I$ has rank one and $A - 5I$ has rank one.

Solution:
This would imply the characteristic polynomial has at least four roots

15) T F

If A is a 3×3 matrix with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 7$, then $A - 3I$ has rank one and $A - 7I$ has rank two.

Solution:
$A - 3I$ could have rank two if the geometric multiplicity of 3 is only one

16) T F

If A is a 10×2 matrix of rank 2, then the least square solution of $Ax = b$ is unique.

Solution:
Yes, this implies that the kernel of A is trivial.

17) T F

If A is an asymptotically stable 3×3 matrix, then A^T is asymptotically stable.

Solution:
A and A^T have the same eigenvalues

18) T F

If a 3×3 matrix A is similar to C and a 3×3 matrix B is similar to D, then AB is similar to CD.

Solution:
One can already find diagonal examples: $A = Diag(2,1), C = Diag(1,2)$ and $B = Diag(3,4), D = Diag(3,4)$, then $AB = Diag(6,4)$ and $CD = Diag(3,8)$.

19) T F

If a 3×3 matrix A is similar to the zero matrix 0 then A is equal to the zero matrix.

Solution:
This is not true for the shear.

20) T F For any 3×3 matrices A, B we know that if A has the same determinant as B^3, then B has the same determinant as A^3.

Solution:
Its already not true for $A = 4I_n$ and $B = 2I_n$.

Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is asymptotically stable? We just write "stable" abbreviating asymptotically stable.

Matrix	stable	not stable
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$		
$\begin{bmatrix} 1/4 & 8 & 8 \\ 0 & 1/4 & 8 \\ 0 & 0 & 1/4 \end{bmatrix}$		

Solution:
The first and last have eigenvalues smaller than one and are therefore stable.

b) (2 points) Which identities hold for a non-invertible 3×3 matrix with eigenvalues α, β, γ and characteristic polynomial $f_A(\lambda)$?

Identity	always true	not always true
$\det(A) = 0$		
$\text{tr}(A) = 0$		
$f_A(\alpha) = 0$		
$\dim(\ker(A)) = 0$		

Solution:

Identity	always true	not always true
$\det(A) = 0$	x	
$\text{tr}(A) = 0$		x
$f_A(\alpha) = 0$	x	
$\dim(\ker(A)) = 0$		x

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z \log(w)}$ and $\log(z) = \ln|z| + i \arg(z)$ with $0 \leq \arg(z) < 2\pi$ for any complex numbers $w \neq 0, z \neq 0$ and where $\arg(z)$ is the angle so that $z = |z|e^{i \arg(z)}$.

Number	is real	is not real
$\log(i\pi)$		
$e^{i\pi}$		
$\log(e^{i\pi})$		
$e^{i\pi/2}$		

Solution:

Number	is real	is not real
$\log(i\pi)$		x
$e^{i\pi}$	x	
$\log(e^{i\pi})$		x
$e^{i\pi/2}$		x

d) (2 points) Which type of matrices are always diagonalizable **over the real numbers**?

Type of matrix	always diagonalizable over the reals	not necessarily diagonalizable
symmetric		
orthogonal		
horizontal shear		
dilation		

Solution:

Type of matrix	always diagonalizable over the reals	not necessarily diagonalizable
symmetric	x	
orthogonal		x
horizontal shear		x
dilation	x	

e) (2 points) If S is a 4×4 matrix whose columns are given by an eigenbasis of a matrix A which has eigenvalues $0, 1, 2, 3$, then

The statement	is always true	can be false
A is invertible		
S is invertible		
$A + I_4$ is invertible		
$A - I_4$ is invertible		

Solution:

The statement	is always true	can be false
A is invertible		x
S is invertible	x	
$A + I_4$ is invertible	x	
$A - I_4$ is invertible		x

Problem 3) (10 points) No justifications are needed

a) (2 points) A, B are arbitrary 3×3 matrices. Each of the two has distinct eigenvalues meaning that the algebraic multiplicity of each eigenvalue is 1. We write $A \sim B$ to indicate that A is similar to B.

The statement	implies $A = B$	implies $A \sim B$
A, B have the same eigenvalues		
A, B have same eigenvectors		
A, B have same eigenvalue,eigenvector pairs		

Solution:

The statement	implies $A = B$	implies $A \sim B$
A, B have the same eigenvalues		x
A, B have same eigenvectors		
A, B have same eigenvalue,eigenvector pairs	x	x

b) (2 points) Assume A is an invertible and diagonalizable 2×2 matrix. Which matrices are diagonalizable too?

A^2	
A^T	
A^{-1}	
$2A + I_2$	

Solution:

A^2	x
A^T	x
A^{-1}	x
$2A + I_2$	x

c) (2 points) Fill in each of the two cases the Q and R matrices giving the QR factorization $A = QR$ of A:

A	Q	R
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$		

Solution:

A	Q	R
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	I	A
$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$	$A/\sqrt{5}$	$\sqrt{5}I$

d) (2 points) Which of the following matrices A has the property that

$$A[x(t), x(t-1), x(t-2)]^T = [x(t) + x(t-1) + x(t-2), x(t), x(t-1)]^T \ .$$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	

Solution:
The second one.

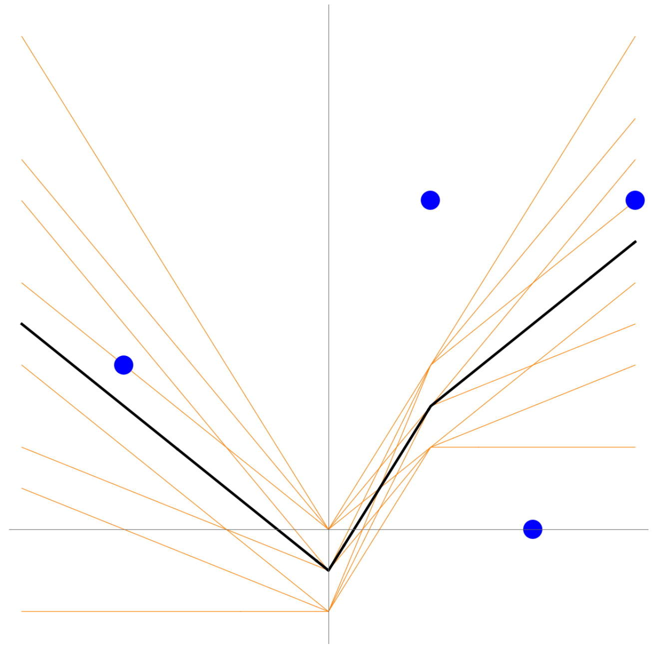
e) (2 points) Exactly one of the three statements is **not** part of the spectral theorem. Which one?

A symmetric matrix has an orthonormal eigenbasis	
A real symmetric matrix has real eigenvalues	
A matrix with real eigenvalues has an orthonormal eigenbasis	

Solution:

A symmetric matrix has an orthonormal eigenbasis	
A real symmetric matrix has real eigenvalues	
A matrix with real eigenvalues has an orthonormal eigenbasis	x

Problem 4) (10 points)



Find the function

$$a|x| - b|x-1| = y$$

which is the best fit for the data

x	y
1	2
3	2
-2	1
2	0

Solution:

Write the data as a system of equations $Ax = b$ with $A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \\ 2 & -3 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

The rest is routine. We have $x = (A^TA)^{-1}A^Tb = [,-1]^T/6$. A few check points: $(A^TA) = \begin{bmatrix} 18 & -14 \\ -14 & 14 \end{bmatrix}$. The inverse is $\begin{bmatrix} 14 & 14 \\ 14 & 18 \end{bmatrix}/56$. $A^Tb = [10,-7]^T$. The best fit is $y = (3/4)|x| - (1/4)|x-1|$.

Problem 5) (10 points)

a) (2 points) Find the determinants of

$$A(2) = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

b) (2 points) Use Laplace expansion to get the determinant of

$$A(4) = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}.$$

c) (2 points) Find the determinant of

$$B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 \end{bmatrix} .$$

d) (2 points) Find the determinant of

$$C = \begin{bmatrix} 5 & 2 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{bmatrix} .$$

e) (2 points) Find the determinant of

$$D = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 & 0 \end{bmatrix} .$$

Solution:
a) 21 and 85 (Direct computation or Laplace expansion)
b) 341 Laplace expansion
c) 5^6 upper triangular
d) 1053 compute eigenvalues
e) -30 just one pattern and count upcrossings.

Problem 6) (10 points)

The recursion $x(t+1) = 5x(t) - 4x(t-1)$ can be written as $v(t+1) = Av(t)$.

a) (3 points) Fill in the 2×2 matrix A to make this a recursion

$$\begin{bmatrix} x(t+1) \\ x(t) \end{bmatrix} = \begin{bmatrix} . & . \\ . & . \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-1) \end{bmatrix} .$$

b) (3 points) Find the eigenvalues and eigenvectors of A.

c) (4 points) Write the initial condition $\vec{v}(0) = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ as a combination of eigenvectors and use this to write down the **closed form solution** for $\vec{v}(t)$.

P.S. The recursion appearing here could have been useful to compute the determinants $A(n)$ in problem 5a) and 5b).

Solution:
The eigenvalues are 1 and 4. The eigenvectors are $[1,1]^T, [4,1]^T$. The initial condition is the sum. The closed form solution is $4^t[4,1]^T + [1,1]$.

Problem 7) (10 points)

The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has the eigenbasis

$$\left\{ \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} .$$

a) (3 points) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A.

b) (3 points) Find the eigenvalues μ_1, μ_2, μ_3 of the inverse

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} .$$

c) (2 points) What are the eigenvectors of A^{-1}?

d) (2 points) Check the boxes which lead to true identities "left expression = expression above":

	tr(A)	det(A)
$\lambda_1 + \lambda_2 + \lambda_3$		
$\lambda_1\lambda_2\lambda_3$		

Solution:
a) Just compute for every eigenvector v, the expression Av. As we know this is a multiple of the eigenvalue, we can read of that factor. For example

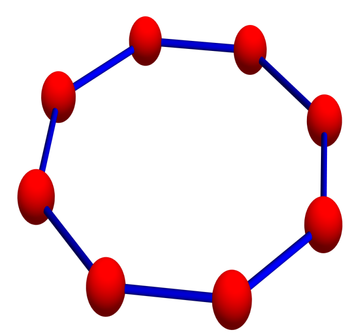
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix} = (1+\sqrt{2}) \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}$$

We see the eigenvalues are $1+\sqrt{2}, 1-\sqrt{2}, 1$.
b) The eigenvectors of A^{-1} are the same, the eigenvalues are $1/(1+\sqrt{2}), 1/(1-\sqrt{2}), 1$. c) The sum of the eigenvalues is the trace, the product of the eigenvalues is the determinant. In this case the sum is 3, the product is -1. In the inverse case, the sum is -1, the product is still -1.
Remark: There is some background story here. Here is a recent theorem in the geometry of finite sets (quantum geometry): *A simplicial complex G is a finite set of n non-empty sets closed under the process of taking non-empty finite subsets. Such a complex defines a $n \times n$ matrix (Laplacian), where $L(x,y) = 1$ if x, y intersect and $L(x,y) = 0$ otherwise. The* **unimodularity theorem** *assures that $\det(L)$ is either 1 or -1 so that the inverse (called Green function) $g = L^{-1}$ has integer entries. In this case, the complex G is given by all nonempty subsets of $\{a,b\}$, so that $G = \{(a),(b),(ab)\}$ has $n = 3$ points. The set (ab) is connected to all leading to the first row of the matrix having 1. The others are only connected to themselves and (a,b).*

Problem 8) (10 points)

The Laplacian of the circular graph C_8 is

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} .$$



The matrix B can be written as $2I - A - A^{-1}$, where A is an orthogonal matrix.

a) (5 points) What are the eigenvalues of B?

b) (5 points) What are the eigenvectors of B?

Solution:
a) Write $B = 2 - A - A^{-1}$ where A is a circular matrix with top row $[0,1,0,0,0,0,0,0]$. The eigenvalues of A are $\lambda_k = e^{2\pi ik/8}$ with $k = 0,1,...,7$. The eigenvalues of B are $\mu_k = 2 - \lambda_k - 1/\lambda_k$.
b) The eigenvectors of A are $v_k = [1, \lambda_k, \lambda_k^2, \dots, \lambda_k^7]^T$.

Problem 9) (10 points)

a) (4 points) Find the QR factorization of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (4 points) Find the QR factorization of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) (2 points) Assume A is an orthogonal 3×3 matrix. What is

$$P = A(A^T A)^{-1} A^T \ ?$$

Solution:
a) A is already orthogonal so that $A = QR = AI$
b) A is already upper triangular so that $A = QR = IA$
c) Simplify $A^T A = I$ and the use $AA^T = I$. The answer is I. Yes this is a projection formula. But in the case when A is invertible, the projection formula gives the projection onto the entire space, which is the identity matrix.

Problem 10) (10 points)

Assume A is an invertible 3×3 matrix.

a) (2 points) Is A^T necessarily invertible?
b) (2 points) Is A^{-1} necessarily invertible?

c) (2 points) Is $A + A^T$ necessarily invertible?
d) (2 points) Is $A + I_3$ necessarily invertible?
e) (2 points) If $A = QR$ is the QR decomposition. Are both R and Q invertible?

Solution:
a) Yes, A has the same determinant than A^T. The determinant determines whether a matrix is invertible. b) Yes, The inverse is A. c) No, Take a rotation by 90 degrees around the z axes. Then $A + A^T$ is not invertible. d) No, Take $A = -I_3$. e) Yes, Q is orthogonal and so invertible. The determinant formula shows that $R = Q^T A$ is invertible.