Name:

MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	A rotation in the plane around the point $(1, 1)$ by angle 90^0 is a linear transformation.
2)	T	F	If $ABC = I_2$ for 2×2 matrices A, B, C, then A is invertible.
3)	T	F	There is a 2×3 matrix A and a 3×2 matrix B such that $AB = BA$.
4)	T	F	For any linear system $Ax = b$ with 3×3 matrix A, the augmented 3×4 matrix $B = [A\|b]$ satisfies rank(A) = rank(B).
5)	T	F	If the system $Ax = b$ has a unique solution for some b, then A must be a square matrix.
6)	T	F	If $v_1, \dots, v_4$ are linearly independent vectors in $\mathbf{R}^4$, then they must form a basis for $\mathbf{R}^4$.
7)	T	F	For every 3×3 matrix A, we have that $\text{rref}(A)$ is similar to A.
8)	T	F	For any two 3×3 matrices A, B the identity $(A+B)(A+B) = A^2+2AB+B^2$ holds.
9)	T	F	The set X of all vectors which are both in the kernel and image of a given 2×2 matrix.
10)	T	F	If A is a non-invertible matrix then $\text{rref}(A)$ has at least one row of zero.
11)	T	F	The plane $2x + 3y + 5z - 10 = 0$ is the image of a linear transformation T.
12)	T	F	For any $n \times n$ matrix A the identity $\ker(A^2) = \ker(\text{rref}\,(A^2))$ holds.
13)	T	F	For any $n \times n$ matrix A, the intersection of the kernel $\ker(A)$ and the image $\text{im}(A)$ is the trivial space $\{0\}$.
14)	T	F	The set of rotation matrices form a linear space.
15)	T	F	A 3×3 matrix A can satisfy $\ker(A) = \text{im}(A)$.
16)	T	F	If the linear system $A\vec{x} = \vec{b}$ has exactly one solution for some vector $\vec{b}$, then it has exactly one solution for all vectors $\vec{b}$.
17)	T	F	If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$, then $\vec{v}_1 \cdot \vec{v}_2 = 0$.
18)	T	F	The rank of an $m \times n$ matrix is at most m.
19)	T	F	If A is a 4×3 matrix and rref(A) has exactly two nonzero rows, then $\dim(\ker(A)) = 1$.
20)	T	F	A reflection in space about a line is similar to a rotation by $90°$ around the z axis.

Total

Problem 2) (10 points) No justifications are needed.

Match each of the following matrices with the geometric descriptions.

Matrix	Enter A-G here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	
d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

A) Projection onto a plane.
B) Rotation around an axis.
C) Reflection at a point.
D) Projection onto a line.
E) Reflection at a plane.
F) Reflection at a line.
G) Identity transformation.

Problem 3) (10 points)

a) (6 points) Given two vectors $X, Y \in R^n$ satisfying $X_1 + ... + X_n = 0$ and $Y_1 + ... + Y_n = 0$. Describe the following statistical terms geometrically:

	standard deviation of X
	correlation of X and Y
	the variance of Y

b) (4 points) In the basis $\mathcal{B} = \{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$, the matrix $A = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$ becomes one of the following two matrices. Check the correct one.

	$B = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$
	$B = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix}$

Problem 4) (10 points)

Consider the system of linear equations

$$\begin{aligned} x + y + z + w &= a \\ 2x + 2y + 2z + 2w &= b \\ 3x + 3y + 3z + 3w &= c\,. \end{aligned}$$

a) (4 points) Are there a, b, c such that the system has exactly one solution? If yes, find such a, b, c. If not, why not?

b) (3 points) Are there a, b, c such that the system has no solution? If yes, find such a, b, c. If not, why not?

c) (3 points) Are there a, b, c such that the system has infinitely many solutions? If yes, find such a, b, c. If not, why not?

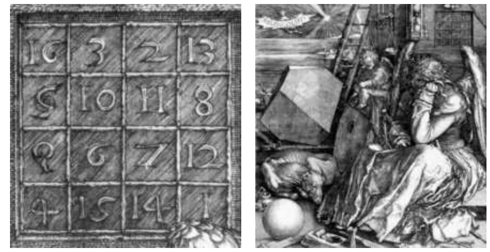
Problem 5) (10 points)

Let A be the matrix which is the reflection about the x-axis in the plane. Let B be the matrix of the transformation which leaves e_1 the same and maps e_2 to $e_1 + e_2$.
Finally, define $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

a) (6 points) Find the product $AB^{100}C$.

b) (4 points) What is $(ABC)^{10}$?

Problem 6) (10 points)



Part of **Melancolia I**, an engraving by the German Renaissance master Albrecht Dürer. This master piece was done in 1514. It contains a latin square which is even a **magic square**: The diagonals add up to the same number too and all matrix elements are different. Additionally, the date 1514 appears in the bottom row of the matrix. By the way: the student in **Melancolia I** takes a linear algebra exam - therefore the name of the picture.

The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is called a **latin square**. The sum of the entries of each row and column vectors add up to the same number 10.

a) (4 points) In the matrix A, the sum of the first and last column is equal to the sum of the second and third column. Find a nonzero vector in the kernel of A.

b) (6 points) Find a basis for the image of A.

Problem 7) (10 points)

Consider the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \vec{d} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

in $\mathbf{R^3}$. Let V be the set of all vectors $\vec{x}$ in $\mathbf{R^3}$ for which all the dot products

$$\vec{a} \cdot \vec{x},\ \vec{b} \cdot \vec{x},\ \vec{c} \cdot \vec{x},\ \vec{d} \cdot \vec{x}$$

are zero.

a) (4 points) Find a 4×3 matrix A whose kernel is V.

b) (4 points) Find a basis of V

c) (2 points) What is the dimension of V?

Problem 8) (10 points)

Let A be a matrix for which

$$B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

a) (6 points) Find a basis for the kernel of A and determine the dimension of the kernel and image of A.

b) (4 points) Is it possible that A was a matrix for which $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ is in the image of A? If yes, give a matrix A. If no, argue why not.

Problem 9) (10 points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

a) (5 points) Find the inverse of A.

b) (5 points) A 3×3 matrix X satisfies the equation

$$AXA = A - I_3$$

Find X.

Problem 10) (10 points)

Let A be a 3×4 and let B be a 4×3 matrix.

a) (3 points) Can the 3×3 matrix AB be invertible?

b) (3 points) Can the 4×4 matrix BA be invertible?

c) (4 points) Find a 3×4 matrix A and a 4×3 matrix B so that AB and BA both have the same rank 2.