Homework 31: Partial differential equations

This is the last homework. It is due on the last day of classes: Wednesday April 25, respectively on Tuesday, April 24, 2018.

1 Solve the heat equation $f_t = 2018 f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = 0$ if $x \in [0, \pi/2]$ and $f(x) = \sin(2x)$ if $x \in [\pi/2, \pi]$.

2 Solve the partial differential equation $u_t = 3u_{xxxxxx} + 5u_{xx}$ with initial condition $u(0, x) = 21x$.

3 A piano string is fixed at the ends $x = 0$ and $x = \pi$ and is initially undisturbed $u(x, 0) = 0$. The piano hammer induces an initial velocity $u_t(x, 0) = g(x)$ onto the string, where $g(x) = \sin(3x)$ on the interval $[0, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$. How does the string amplitude $u(x, t)$ move, if it follows the wave equation $u_{tt} = u_{xx}$?

4 A laundry line is excited by the wind. It satisfies the differential equation $u_{tt} = u_{xx} + \cos(t) + \cos(3t)$. Assume that the amplitude u satisfies initial position $u(x, 0) = x$ and $u_t(x, 0) = 4\sin(5x) + 10\sin(6x)$. Find the function $u(x, t)$ which satisfies the differential equation.
Hint. First find the general homogeneous solution $u_{homogeneous}$ of $u_{tt} = u_{xx}$ for an odd u then a particular solution $u_{particular}(t)$ which only depends on t. Finally fix the Fourier coefficients.

5 We have looked at 4 different types of differential equations. Systems of linear differential equations $x' = Ax$, nonlinear equations $x' = f(x, y), y' = g(x, y)$, inhomogeneous equations $p(D)f = g$ as well as partial differential equations like $u_t = D^2u$ and the wave equation $u_{tt} = D^2u$. Give an original example of each type (it should not have appeared in any homework nor handout).

Partial differential equations

Solving a PDE means to find a function f on the interval $[0, \pi]$. We write it as a sin-series which means that we only need to compute the b_n using the formula

$$\frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx \, .$$

This is justified as we can think of f continued as $f(-x) = -f(x)$ on $[-\pi, 0]$ The temperature distribution $f(x,t)$ in a metal bar $[0, \pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t) = D^2 f(x,t)$$

Here μ is a positive constant which depends on the material. The height of a string $f(x,t)$ at time t and position x on $[0, \pi]$ satisfies the **wave equation**

$$f_{tt}(x,t) = c^2 f_{xx}(x,t) = c^2 D^2(f)(x,t)$$

Here c is a positive constant which tell how fast the waves move. All problems are solved by diagonalizing D^2 using a Fourier basis. For heat, write the initial condition as a Fourier series and write down the solution. For wave, write both the initial condition $f(0,x)$ as well as the initial velocity $f_t(0,x)$ as a Fourier series and write down the solutions.