Homework 30: Fourier II

This homework is due on Friday, April 20, respectively on Tuesday, April 24, 2018.

1 Find the Fourier series of the function which is 8 on $[0, \pi/4]$ and zero everywhere else.

Solution:
The function is neither even nor odd so that we have to compute all Fourier coefficients. In all cases, just integrate from 0 to $\pi/4$ because the function is only nonzero there $a_0 = \frac{1}{\pi} \int_0^{\pi/4} 8 \cdot 1/\sqrt{2}\, dx = 8/(4\sqrt{2})$.
$b_n = \frac{1}{\pi} \int_0^{\pi/4} 8 \cdot \sin(nx)\, dx = -8\cos(n\pi/4)/(n\pi)$
$a_n = \frac{1}{\pi} \int_0^{\pi/4} 8 \cdot \cos(nx)\, dx = 8(\sin(n\pi/4) - 1)/(n\pi)$

2 Use Parseval to find the length $|f| = \sqrt{(1/\pi) \int_{-\pi}^{\pi} f(x)^2\, dx}$ for $f(x) = 2\cos(14x) + 4\cos(11x) + 2\sin(27x) - \cos(19x) + 5\cos(140x)$.

Solution:
Parseval allows to express this as a square root of the sum of squares of the Fourier coefficients. $\sqrt{(2^2 + 4^2 + 2^2 + 1^2 + 5^2)} = \sqrt{50}$

3 Compute both sides of the Parseval identity for $f(x) = x + |x|$.

Solution:
The function is $f(x) = 2x$ if $x > 0$ and $f(x) = 0$ if $x < 0$. We have $||f||^2 = \frac{1}{\pi}\int_0^\pi 4x^2 \; dx = 4\pi^2/3$. This is the left hand side of Parseval. The function is neither odd nor even, so that we have to compute both a_n and b_n. We have $a_0 = (1/\pi)\int_0^\pi 2x1/\sqrt{2} \; dx = \pi/\sqrt{2}$. $a_n = (1/\pi)\int_0^\pi 2x\cos(nx) \; dx = (2/n^2)(\cos(n\pi) - 1)$ and $b_n = (1/\pi)\int_0^{pi} 2x\sin(nx) \; dx = (2/n^2)(-n\pi\cos(n\pi))$. We have $a_0^2 + \Sigma_n a_n^2 + \Sigma_n b_n^2 = \pi^2/2 + \Sigma_n[(2/n^2)(\cos(n\pi) - 1)]^2 + [(2/n^2)(-n\cos(n\pi))]^2$.

4 Find $\Sigma_{n=1}^\infty \frac{1}{(2n)^2} = 1/4 + 1/16 + 1/36 + \dots$ from the known Basel problem formula of $\Sigma_n \frac{1}{n^2}$ and use this to compute the sum $\Sigma_{n=0}^\infty \frac{1}{(2n+1)^2}$ over the odd numbers.

Solution:
We know that $\Sigma_n 1/n^2 = \pi^2/6$. Therefore, $\Sigma_n 1/(2n)^2 = \pi^2/6(\frac{1}{4})$. Because $\Sigma_n 1/(2n+1)^2 = \Sigma_n 1/n^2 - 1/(2n)^2$, we have $\Sigma_n 1/(2n+1)^2 = \pi^2/6 - \pi^2/6(\frac{1}{4}) = \pi^2/8$.

5 This problem is a preparation for partial differential equations PDEs and consists of reminders. All statements are pretty straight forward. We work with functions on $[-\pi, \pi]$ described by Fourier series. You can refer to work done in the previous homework:
a) Verify that the Fourier basis $\mathcal{B} = \{1/\sqrt{2}, \cos(nx), \sin(nx)\}$ consists of eigenfunctions of D^2 on the space of piecewise smooth 2π-periodic functions.
b) What are the corresponding eigenvalues?
c) Show that every eigenfunction of D^2 is either constant or of the form $a\cos(nx) + b\sin(nx)$ for some n.
d) What are the eigenvalues of $D^2 + D^4 + 6$ on the subspace of

C^{∞}_{per} consisting of odd functions?

Solution:
a) Just check that $D^2 1 = 0 \cdot 1$ and $D^2 \cos(nx) = (-n^2)\cos(nx)$ or $D^2 \sin(nx) = (-n^2)\sin(nx)$ by differentiation.
b) The eigenvalues are $-n^2$, which includes $n = 0$.
c) We can either cite that we know that $1/\sqrt{2}, \sin(nx), \cos(nx)$ is an orthonormal eigenbasis or then redo $(D^2+n^2)f = 0$ factors as $(D-in)(D+in)f = 0$ giving the solution space spanned by e^{inx}, e^{-inx}. This is the same solution space (over the complex numbers) as the one spanned by $\cos(nx), \sin(nx)$. The reason is that $e^{inx} = \cos(nx) + i\sin(nx)$ and $e^{-inx} = \cos(nx) - i\sin(nx)$.
d) Since $D^2 \sin(nx) = (-n^2)\sin(nx)$, we have $D^4 \sin(nx) = n^4 \sin(nx)$ and $D^2 + D^4 + 6$ has the eigenvalues $n^4 - n^2 + 6$ for $n = 1, 2, 3, \dots$.

Fourier Series II

Recall that the Fourier coefficients of a function $f \in C^\infty_{per}$ are defined as $a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\ dx$, $a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$, $b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx$. These are just inner products $a_n = \langle f, \cos(nx) \rangle$ and $b_n = \langle f, \sin(nx) \rangle$ and

$$f = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx)$$

is the Fourier series of f. The Parseval identity is $||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2$. It is an extension of the Pythagoras theorem.