Homework 29: Fourier I

This homework is due on Wednesday, April 18, respectively on Thursday, April 19, 2018.

1 a) Find the angle between the functions $f(x) = x^2$ and $g(x) = x^3$.
b) Project $f(x) = \sin^2(x)$ onto the plane spanned by $\sin(2x), \cos(2x)$.
c) Find the length of the function $f(x) = x^3$ in C^∞_{per}.

Solution:
a) We compute the inner product $\langle f, g \rangle = \pi^{-1} \int_{-\pi}^{\pi} x^5 \, dx = 0$. Therefore the angle is $\pi/2$, as the two functions are perpendicular to each other.
b) Because $\sin(2x)$ and $\cos(2x)$ are orthogonal, the projection is given by the function

$$\langle \sin^2(x), \cos(2x) \rangle \cos(2x) + \langle \sin^2(x), \sin(2x) \rangle \sin(2x) \ .$$

We get $(1/\pi) \int_{-\pi}^{\pi} \sin^2(x) \sin(2x) \quad dx = 0$ and $(1/\pi) \int_{-\pi}^{\pi} \sin^2(x) \cos(2x) \quad dx = -1/2$ so that the projection is $-\cos(2x)/2$.
c) $|f| = \sqrt{\langle f, f \rangle} = \sqrt{\pi^{-1} \int_{-\pi}^{\pi} x^3 \, dx} = \sqrt{(\pi)^{-1}(2\pi^3/4)} = \sqrt{2\pi^2/4}$.

2 Verify that the functions $\cos(nx), \sin(nx), 1/\sqrt{2}$ form an orthonormal family.

Solution:
We have to check a couple of cases, like $\langle \cos(nx), \cos(mx)\rangle, \langle \sin(nx), \sin(mx)\rangle, \langle \cos(nx), \sin(mx)\rangle$, $\langle 1/\sqrt{2}, \cos(nx)\rangle, \langle 1/\sqrt{2}, \sin(nx)\rangle$ and then also check that all vectors have length one. We use the identities provided. For example, $\langle \sin(nx), \sin(mx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos((n-m)x) - \cos((n+m)x)\ dx$ which is equal to 1 if $n = m$ and equal to 0 if $n \neq m$. The computations can be abbreviated by noting that integrating an odd 2π-periodic function over $[-\pi, \pi]$ is zero because the integral on $[0, \pi]$ cancels with the integral on $[-\pi, 0]$.

3 Find the Fourier series of the function $f(x) = 11 + |6x|$.

Solution:
First get the Fourier series of the function $g(x) = |6x|$ and then deal with the constants. This is an **even function** so it has a cosine only series. We compute

$$a_0 = \langle g, 1/\sqrt{2} \rangle = 6\frac{2}{\pi}\int_0^\pi x\frac{1}{\sqrt{2}}\,dx = 6\frac{\pi\sqrt{2}}{2}\,.$$

$$a_n = \langle g, \cos(nx) \rangle = \frac{2}{\pi}\int_0^\pi 6x\cos(nx)\,dx = 6\frac{2}{\pi}[\frac{\cos(n\pi)-1}{n^2}]\,.$$

The Fourier coefficients of the constant function 11 is $a_0 = 11\sqrt{2}$ because $a_0 1/\sqrt{2} = 11$. All the other coefficients are zero. The Fourier coefficients of $f(x) = 11 + |6x|$ is then the sum:

$$a_0 = 11\sqrt{2} + 6\frac{\pi\sqrt{2}}{2}$$

and

$$a_n = 6\frac{6}{\pi}[\frac{\cos(n\pi)-1}{n^2}]\,.$$

4 Find the Fourier series of the function $4\cos(3x) + \sin^2(13x) + \sin(100x) + 10$.

Solution:
The Fourier series of $4\cos(3x) + \sin(100x)$ is $4\cos(3x) + \sin(100x)$. Using the double angle formulas already establishes the Fourier series $(1 - \sin(26x))/2$. In total we have $4\cos(3x) + 9/2 - (1/2)\sin(26x)$.
Important remark: if a function is a trig function, it is already its Fourier series. For example, to get the Fourier series of the function $f(x) = 5\sin(3x)$, we have the answer $5\sin(3x)$ because all terms $\langle f, \cos(nx)\rangle$ are zero, and all terms $\langle f, \sin(mx)\rangle$ with m different from 3 are zero (they are perpendicular as you have shown!) The only Fourier coefficient which is left is $b_3 = \langle f, \sin(3x)\rangle = \langle 5\sin(3x), \sin(3x)\rangle = 5$. So that the Fourier sum is just $5\sin(3x)$.

5 Find the Fourier series of the function $f(x) = 10|\sin(x)|$.

Solution:
To find the Fourier series $a_0 1/\sqrt{2} + \Sigma_{n\geq 1}\, a_n \cos(nx)$ of the function $f(x) = 10|\sin(x)|$, we first note that this is an **even function** so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $10\sin(x)$ instead of $10|\sin(x)|$ so that

$$a_0 = \frac{20}{\pi}\int_0^\pi \sin(x)/\sqrt{2}\, dx = \frac{20\sqrt{2}}{\pi}$$

$$a_n = \frac{20}{\pi}\int_0^\pi \sin(x)\cos(nx)\, dx = \frac{40}{\pi}\frac{1}{1-n^2}$$

for even n and $a_n = 0$ for odd n.
To do the integral, we used the trigonometric identities $2\sin(x)\cos(nx) = \sin(x + nx) + \sin(x - nx)$.

Fourier Series I

We write 2π periodic functions always as functions on $[-\pi, \pi]$. In the space of piecewise smooth periodic functions C^∞_{per} we have the inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$. This defines the **length** $|f| = \sqrt{\langle f, f \rangle}$ and angle $\cos(\alpha) = \langle f, g \rangle / (|f||g|)$. The functions $\sin(nx), \cos(nx), 1/\sqrt{2}$ form an orthonormal basis in C^∞_{per} in the sense that they are linearly independent and span the space. Any function f can be written uniquely as a linear combination of these terms. With the Fourier coefficients $a_0 = \langle f, 1/\sqrt{2} \rangle, a_n = \langle f, \cos(nx) \rangle$ and $b_n = \langle f, \sin(nx) \rangle$, we can write

$$f = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + \sum_n b_n \sin(nx)$$

This is called the Fourier series of f.

Example: Find the length of the function $f(x) = x^2$.
Solution: since

$$||f||^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4\, dx = 2\pi^4/5 \ ,$$

we have $||f|| = \sqrt{2/5}\pi^2$.

Example: find the Fourier series of the function $f(x) = x^2$ on $[-\pi, \pi]$.

Solution: the function is an even function. So, all the coefficients b_n are zero. We compute $a_0 = \frac{2}{\pi}\int_0^\pi x^2 1/\sqrt{2}\ dx = \sqrt{2}\pi^2/3$ and $a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos(nx)\ dx = (4\cos(n\pi))/n^2 = 4(-1)^n/n^2$. The Fourier series is

$$a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \pi^2/3 + \sum_{n=1}^{\infty} 4\cos(n\pi)/n^2 \cos(nx)\ .$$

Below you see the sum with 3 terms. It already gives quite a good fit of the parabola which is continued to a periodic function.