Homework 28: Differential equations

This homework is due on Monday, April 16, respectively on Tuesday, April 17, 2018.

1 a) Find the general solution to $f' = 30\log(t) + 20e^{-5t}$.
b) Find the general solution to $f'' = 1/\sqrt{1+t^2}$.

Solution:
a) We just integrate

$f(t) = \int 30\log(t) + 20e^{-5t} dt = 30(t\log(t) - t) - 20e^{-5t}/5 + C$.

b) We integrate to get $f' = \arcsin(t) + c_1$, then integrate again to get $\sqrt{1-t^2} + t\arcsin(t) + c_1 t + c_2$.

2 a) Find the solution to $f' - 5f = e^{7t}$ by inverting the operator $D - 5$.
b) Find the solution again, this time with the cookbook: find first the solution of $f' - 5f = 0$, then make a guess for a special solution.

Solution:
a) We have $\lambda = 5$, so

$$f = Ce^{5t} + e^{5t}\int_0^t e^{-5s}e^{7s} ds = Ce^{5t} + e^{7t}/2$$

b) The homogeneous solution is $f = Ce^{5t}$. For the special solution, try Ae^{7t}. Plugging it into the equation gives $A = 1/2$. Thus the general solutions are $\frac{1}{2}e^{7t} + Ce^{5t}$.

3 a) Find the general solution of $f'' + 225f = e^{44t} + t$.
b) Find the general solution of $f'' + 225f = \cos(15t)$.
c) Find the general solution of $f'' + 225f = \cos(10t)$.
d) Why does the solution in b) behave differently than the solution in c).

Solution:
The homogeneous equation $f'' + 225f = 0$ in all cases can be solved by $(D^2 + 225)f = (D + 15i)(D - 15i)f = 0$. Thus the solutions are $c_1 e^{15it} + c_2 e^{-15it}$, or equivalently $C_1\cos(15t) + C_2\sin(15t)$ with some other constants C_1, C_2.
a) To find a special solution, we try $Ae^{44t} + Bt + C$. The solution is $C_1\cos(15t) + C_2\sin(15t) + e^{44t}/2161 + t/225$.
b) To find a special solution, try $A\cos(15t)$. Does not work. We have to put a t and need to try $At\cos(15t) + Bt\sin(15t)$. The solution is $C_1\cos(15t) + C_2\sin(15t) + t\sin(15t)/225$.
c) To find a special solution, try $A\cos(10t)$ and get A. $C_1\cos(15t) + C_2\sin(15t) + \cos(10t)/125$.

4 a) Find the general solution to $f'' - 10f' + 25f = 500t$.
b) Find the general solution to $f'' - 10f' + 25f = 4e^{5t}$

Solution:
The homogeneous equation $f'' - 10f' + 25f = 0$ can be solved by $(D^2 - 10D + 25)f = (D-5)^2 f = 0$. Because $D-5$ appears twice, the solutions are $C_1 te^{5t} + C_2 e^{5t}$.
a) Plugging in $Ax + B$ gives the special solution $8 + 20x$.
b) We can not use e^{5t} nor te^{5t}. Plugging in At^2e^{5t} gives the special solution $2e^{5t}t^2$. The general solution is $C_1 te^{5t} + C_2 e^{5t} + 2e^{5t}t^2$.

5 a) Find the general solution to $f''' - 2f'' - f' + 2f = \sin(3t)$.
b) Find the general solution to $f''' - 2f'' - f' + 2f = e^t + e^{-t}$.

Solution:
The homogeneous equation $f''' - 2f'' - f' + 2f = 0$ can be solved by $(D^3 - 2D^2 - D + 2)f = (D-2)(D-1)(D+1)f = 0$. The solutions are $C_1 e^{2t} + C_2 e^t + C_3 e^{-t}$.
a) We have $(D^3 - 2D^2 - D + 2)(\sin(3t)) = 20\sin(3t) - 30\cos(3t)$, and $(D^3 - 2D^2 - D + 2)(\cos(3t)) = 30\sin(3t) + 20\cos(3t)$. A solution can then be $\frac{2}{130}\sin(3t) + \frac{3}{130}\cos(3t)$, and the general solution is $\frac{1}{65}\sin(3t) + \frac{3}{130}\cos(3t) + C_1 e^{2t} + C_2 e^t + C_3 e^{-t}$.
b) Because e^t and e^{-t} already appear as solutions to the homogeneous equation, our first guesses are te^t and te^{-t}. We have $(D^3 - 2D^2 - D + 2)(te^t) = -2e^t$ and $(D^3 - 2D^2 - D + 2)(te^{-t}) = 6e^{-t}$. Thus $-\frac{1}{2}te^t + \frac{1}{6}te^{-t}$ is one solution and the general solution is $-\frac{1}{2}te^t + \frac{1}{6}te^{-t} + C_1 e^{2t} + C_2 e^t + C_3 e^{-t}$.

Differential equations

To solve a differential equation $(D-3)f = 0$ we write it as $f' = 3f$ and know that $f = Ce^{3t}$ is a solution, where C is a constant. In this section, we use the variable t and write $f(t)$ as $f(t)$ often represents a quantity which changes in time. We write $f'(t) = d/dt f(t)$ or Df to simplify writing. To solve a differential equation like $(D^2 + 2D - 15)f = 0$, we factor the polynomial and write $(D-3)(D+5)f = 0$ which has solutions satisfying $(D+5)f = 0$ and $(D-3)f = 0$. The linear space of solutions to $(D-3)(D+5)f = 0$ is therefore given by functions $f(t) = Ae^{-5t} + Be^{3t}$, where A, B are constants. To find the solution to the inhomogeneous equation like $(D-3)f = e^t$, we can invert the operator $D - \lambda = D - 3$. Its inverse is explicitly given by

$$f(t) = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds \ .$$

In the case $g(t) = e^t$, this gives $f(t) = Ce^{3t} + e^t/2$. To solve $(D-3)(D+5)f = e^t$, we have now to solve $(D+5)f = Ce^{3t} + e^t/2$ and therefore invert $D+5$ with the above formula. We get the general solution $f(t) = C_1 e^{3t} + C_2 e^{-5t} + (-1/12)e^t$. This general operator method works universally, but it can be a bit tedious. In a handout, we show how to get a special solution using a "cookbook recipe". This is how engineers solve the systems.