Homework 27: Differential operators

This homework is due on Friday, April 13, respectively on Tuesday, April 17, 2018.

The linear spaces C^∞, C^∞_{per}, P and T are defined on the next page.

1 The linear map $Df(x) = f'(x)$ is an example of a **differential operator**. It has the constant functions as the kernel. This means that there is no unique inverse. One inverse is $Sf(x) = D^{-1}f(x) = \int_0^x f(t)\,dt$.
a) Evaluate $D\sin$, $D\cos$, $D\tan$, $S1/(1+x^2)$, $S\tan$.
b) Can you find an eigenfunction (= eigenvector) f of D to the eigenvalue -101?
c) Verify that if f is an eigenfunction of D to the eigenvalue 2, then f is also an eigenfunction of $D^4 - 2D + 77$. What is the eigenvalue?

2 a) Find a solution of the equation $D^2 f = 2x + 1/x$ on the space $C^\infty((0,\infty))$ of all smooth functions on the positive real axes.
b) Find two linearly independent solutions of the eigenvalue equation $D^2 f = -10'000 f$ on the space C^∞_{per}

3 a) Find a basis for the kernel of D^3 on the linear space P of polynomials.
b) Find the image $D^3 + D + 1$ on the linear space P?
c) Find the eigenvalues of $D^3 + D + 1$ on the space C^∞_{per} of smooth periodic functions with period 2π.
d) Find the kernel of $Af = (D - \sin(t))f(t)$ on C^∞_{per}.

4 a) Check that $Qf(x) = xf(x)$ and $Pf(x) = iDf(x)$ satisfy the Heisenberg commutation relation: $(PQ - QP)f = if$.
b) Check that for any real ω, the function $e^{i\omega t}$ is an eigenfunction of iD in C^∞.
c) Check that on C^∞_{per}, only the functions $e^{i\omega t}$ with integer ω are eigenfunctions. (Momentum ω is quantized.)

5 a) Verify that $Sf(x) = \int_0^x f(t)\ dt$ is a linear operator on the linear space C^∞ of smooth functions.
b) Show that $DSf(x) = f(x)$ and c) show that $SDf(x) = f(x) - f(0)$. What is the theorem?

Differential operators

A function is **smooth** if it can be differentiated arbitrarily often. The space C^∞ of real valued **smooth functions** is a linear space: if f, g are in C^∞, then $f + g$, the zero function 0 is in C^∞ and λf is in C^∞ for every real λ. C^∞ contains the linear space P of all **polynomials**. The space C^∞_{per} of smooth periodic functions with period 2π forms a linear space too. It contains the linear subspace T of **trigonometric polynomials**. The space P of **polynomials** is spanned by $\{1, x, x^2, x^3, ...\}$ and the space T of trigonometric polynomials is spanned by $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), ...\}$. They are infinite dimensional. The space P_3 of cubic polynomials $d + cx + bx^2 + ax^3$ is 4-dimensional as it has the basis $\{1, x, x^2, x^3\}$. The transformation map $D : f \to f'$ is linear: it satisfies $D(f + g) = Df + Dg, D(\lambda f) = \lambda Df$ and $D0 = 0$. We call any polynomial of D like $D^2 - D + 1$ a **differential operator**. The linear map D on C^∞ has as the kernel the one dimensional space of constant functions. What are the eigenvalues and eigenvectors of D? Because $De^{\lambda x} = \lambda e^{\lambda x}$, every real number λ is an eigenvalue on C^∞. The linear map D has no real eigenvalues on C^∞_{per} but complex eigenvalues in as $De^{inx} = ine^{inx}$, where n is an integer. The fact that they are quantized is the reason why quantum mechanics is called "quantum" (the operator $P = iD$ is called "momentum") and $Qf = xf$ "position"). The square $-D^2$ has now real eigenvalues n^2, where n is an integer. It is the energy operator of a particle on the circle. The eigenfunctions are $1, \sin(nx)$ and $\cos(nx)$. We are interested in D because it will allow us to solve differential equations like $(D^2 + 5D + 6)f = \sin(5x)$.