Homework 27: Differential operators

This homework is due on Friday, April 13, respectively on Tuesday, April 17, 2018.

The linear spaces C^∞, C^∞_{per}, P and T are defined on the next page.

1 The linear map $Df(x) = f'(x)$ is an example of a **differential operator**. It has the constant functions as the kernel. This means that there is no unique inverse. One inverse is $Sf(x) = D^{-1}f(x) = \int_0^x f(t)\,dt$.
a) Evaluate $D\sin, D\cos$, $D\tan$, $S1/(1+x^2)$, $S\tan$.
b) Can you find an eigenfunction (= eigenvector) f of D to the eigenvalue -101?
c) Verify that if f is an eigenfunction of D to the eigenvalue 2, then f is also an eigenfunction of $D^4 - 2D + 77$. What is the eigenvalue?

Solution:
a) We have $D\sin(x) = \cos(x), D\cos(x) = -\sin(x), D\tan(x) = \frac{1}{1+x^2}$, and $S\frac{1}{1+x^2} = \tan(x)$.
b) If f exists, it satisfies $Df = -101f$, so $f'(x) = -101f(x)$. We know this to be the differential equation defining the exponential function $f(x) = \exp(-101x)$.
c) Because $Df = 2f$, we can apply D repeatedly to obtain $D^4 f = 2^4 f, -2Df = -4f$. Thus, $(D^4 - 2D + 77)f = (16 - 4 + 77)f = 89f$, so f is an eigenfunction of the operator with eigenvalue 89.

2 a) Find a solution of the equation $D^2 f = 2x + 1/x$ on the space $C^\infty((0,\infty))$ of all smooth functions on the positive real axes.
b) Find two linearly independent solutions of the eigenvalue equation $D^2 f = -10'000 f$ on the space C^∞_{per}

Solution:
a) We can integrate twice to obtain $Df(x) = 2x + \ln x + a$, and then $f(x) = x^2 x \ln x - x + ax + b$. Setting $a = 1, b = 0$, we obtain one such example of $f(x) = x^2 + x \ln x$.
b) We can rewrite the equation as $(D^2 + 10'000)f = 0$, so any eigenfunction of D with eigenvalue $\pm 100i$ is an eigenfunction of D^2. In particular, $\exp(100ix), \exp(-100ix)$ are eigenfunctions of D^2 with eigenvalue $-10'000$. If we want real eigenfunctions, then we can take the sum and difference to get $\cos(100x), \sin(100x)$ as our pair of linearly independent eigenfunctions of D^2.

3 a) Find a basis for the kernel of D^3 on the linear space P of polynomials.
b) Find the image $D^3 + D + 1$ on the linear space P?
c) Find the eigenvalues of $D^3 + D + 1$ on the space C^∞_{per} of smooth periodic functions with period 2π.
d) Find the kernel of $Af = (D - \sin(t))f(t)$ on C^∞_{per}.

Solution:
a) We can solve the differential equation $D^3 f = 0$ by integrating 3 times and obtaining the polynomials $\{a + bx + cx^2\}$. An obvious basis for the solutions is $\{1, x, x^2\}$.
b) The differential equation $f''' + f' + f = g$ can be solved for f in any case. In other words, $\text{im}(D^3 + D + 1) = P$. We will see this in the next lecture explicitly.
c) The eigenvalues of D are $i \cdot n$, for n an integer. The eigenvalues of D^3 are $(in)^3$. The eigenvalue of $D^3 + D + 1$ are $(in)^3 + in + 1 = 1 + i(n - n^3)$.
d) Solve $Af = 0$ which means $f' - \sin(t)f = 0$. We see either $f = 0$, or $df/f = \sin(t)dt$ so that $\ln|f| = -\cos(t) + c$ and therefore $f = \pm\exp(-\cos(t) + c) = \pm e^c \exp(-\cos(t))$. We can rewrite this as $f = C\exp(-\cos(t))$ for some constant C. That is, $\ker(A) = \{C\exp(-\cos(t))\}$.

4 a) Check that $Qf(x) = xf(x)$ and $Pf(x) = iDf(x)$ satisfy the Heisenberg commutation relation: $(PQ - QP)f = if$.
b) Check that for any real ω, the function $e^{i\omega t}$ is an eigenfunction of iD in C^∞.
c) Check that on C^∞_{per}, only the functions $e^{i\omega t}$ with integer ω are eigenfunctions. (Momentum ω is quantized.)

Solution:
a) Use the product rule: $PQf = i(xf)' = if + ixf'$ and $QPf = xif' = ixf'$. Taking the difference gives $(PQ - QP)f = if$.
b) This is simply $(iD)e^{i\omega t} = i \cdot \frac{d}{dt}e^{i\omega t} = -\omega e^{i\omega t}$, that is, $e^{i\omega t}$ is an eigenfunction associated to eigenvalue $-\omega$.
c) On one hand, $(iD)f = -\omega f \Rightarrow f = Ce^{i\omega t}$, that is, only functions of the form $e^{i\omega t}$ can be eigenfunctions. On the other hand, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and thus the function $e^{i\omega t}$ has period 2π only when ω is an integer; when t changes by 2π, ωt changes by $2\omega\pi$.

5 a) Verify that $Sf(x) = \int_0^x f(t)\ dt$ is a linear operator on the linear space C^∞ of smooth functions.
b) Show that $DSf(x) = f(x)$ and c) show that $SDf(x) = f(x) - f(0)$. What is the theorem?

Solution:
a) We must check three things: $S(f+g) = Sf + Sg$ and $S\lambda f = \lambda Sf$ and $S0 = 0$. But these are all basic properties of the integral.
b) Differentiate the integral $\int_0^x f(t)\ dt$ with respect to x gives $f(x)$.
c) Integrating the derivative $\int_0^x f'(t)\ dt$ gives $f(x) - f(0)$.
The statements b) and c) together are the fundamental theorem of calculus.

Differential operators

A function is **smooth** if it can be differentiated arbitrarily often. The space C^∞ of real valued **smooth functions** is a linear space: if f, g are in C^∞, then $f + g$, the zero function 0 is in C^∞ and λf is in C^∞ for every real λ. C^∞ contains the linear space P of all **polynomials**. The space C^∞_{per} of smooth periodic functions with period 2π forms a linear space too. It contains the linear subspace T of **trigonometric polynomials**. The space P of **polynomials** is spanned by $\{1, x, x^2, x^3, ...\}$ and the space T of trigonometric polynomials is spanned by $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), ...\}$. They are infinite dimensional. The space P_3 of cubic polynomials $d + cx + bx^2 + ax^3$ is 4-dimensional as it has the basis $\{1, x, x^2, x^3\}$. The transformation map $D : f \to f'$ is linear: it satisfies $D(f + g) = Df + Dg, D(\lambda f) = \lambda Df$ and $D0 = 0$. We call any polynomial of D like $D^2 - D + 1$ a **differential operator**. The linear map D on C^∞ has as the kernel the one dimensional space of constant functions. What are the eigenvalues and eigenvectors of D? Because $De^{\lambda x} = \lambda e^{\lambda x}$, every real number λ is an eigenvalue on C^∞. The linear map D has no real eigenvalues on C^∞_{per} but complex eigenvalues in as $De^{inx} = ine^{inx}$, where n is an integer. The fact that they are quantized is the reason why quantum mechanics is called "quantum" (the operator $P = iD$ is called "momentum") and $Qf = xf$ "position"). The square $-D^2$ has now real eigenvalues n^2, where n is an integer. It is the energy operator of a particle on the circle. The eigenfunctions are $1, \sin(nx)$ and $\cos(nx)$. We are interested in D because it will allow us to solve differential equations like $(D^2 + 5D + 6)f = \sin(5x)$.