Homework 26: Nonlinear systems

This homework is due on Wednesday, April 11, respectively on Thursday, April 12, 2018. There is a handout for this material on the website.

1 Analyze the system

$$\begin{aligned} \frac{dx}{dt} &= 2x - x^2 + xy \\ \frac{dy}{dt} &= 4y - xy - y^2 \end{aligned}$$

It is an interaction model of species so that we only look at $x \geq 0, y \geq 0$.

2 We analyze the system

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + ky - k) \\ \frac{dy}{dt} &= y(1 - y + kx - k) \end{aligned}$$

in the cases $k = 2$ and $k = 0$ as well as $k = -2$. Again, as this is a population model, we only look at $x \geq 0, y \geq 0$.

3 Analyze the frictionless pendulum

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2\sin(x) \ , \end{aligned}$$

4 Analyze the system

$$\begin{aligned} \frac{dx}{dt} &= x^2 + y^2 - 1 \\ \frac{dy}{dt} &= xy \end{aligned}$$

5 Analyze the pendulum with friction

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\sin(x) - y\ .\end{aligned}$$

Nonlinear systems

Differential equations $x' = f(x,y), y' = g(x,y)$ generalize the linear case $x' = ax + by, y' = cx + dy$. To analyze such systems when f, g are not linear, we draw **phase portraits**. The curves where $f(x,y) = 0$ or $g(x,y) = 0$ are called nullclines. They intersect in **equilibrium points**. These are points where $x' = 0, y' = 0$. We can use linear algebra to analyze the system near such an equilibrium point (a,b). The matrix $A = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$ is called the **Jacobian matrix**. The linear system $v' = Av$ is called the **linearization** at (x_0, y_0). If this linear system is stable, the equilibrium point is stable. In terms of the original nonlinear system, an equilibrium point (x_0, y_0) is stable if all trajectories starting sufficiently close to (x_0, y_0) tend to it as $t \to \infty$.

Making an **analysis** of the system consists of 1) finding the nullclines and equilibria 2) determine the stability of the equilibria 3) drawing the phase portrait of the system 4) analyzing the possible behaviors of the trajectories.