

Homework 24: Differential equations I

This homework is due on Friday, April 6, respectively on Tuesday, April 10, 2018.

- 1 a) Solve the differential equation $\frac{dx}{dt} = t/(5x^4)$, with $x(0) = 2$.
 b) Solve the differential equation $\frac{dx}{dt} = 1 + x^2$, with $x(0) = 0$.
 c) Solve the differential equation $\frac{dx}{dt} = 1/\cos(x)$, with $x(0) = 0$.

- 2 Solve the system

$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix}$$

with initial condition $x(0) = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$.

- 3 a) For which real p, q is the system $\frac{dx}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} x(t)$ stable? b)

For which real p is the system $\frac{dx}{dt} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} x(t)$ stable? c) For

which a is the system $\frac{dx}{dt} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} x(t)$ stable?

- 4 The interaction of two animal species is modeled by the equations

$$\begin{aligned} \frac{dx}{dt} &= 1.5x - 1.2y \\ \frac{dy}{dt} &= 0.8x - 1.4y \end{aligned}$$

- a) Interpret the system. Is it a symbiosis, competition or predator-prey?
 b) Sketch the phase portrait in the first quadrant.
 c) What happens in the long term? Does it depend on the initial population? If so, how?

- 5 A door opens on one side only. A spring mechanism closes the door which forms an angle $\theta(t)$ with the frame. The angular velocity is $\omega(t) = \frac{d\theta}{dt}(t)$. The differential equations are

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -2\theta - 3\omega\end{aligned}$$

The first equation is the definition, the second incorporates the force -2θ of the spring and the friction -3ω .

Sketch a phase portrait for the system and use this to answer the question, for which initial conditions, the door slams (reaches $\theta = 0$ with negative ω).

Differential Equations I

$\frac{dx}{dt} = f(x)$ is a differential equation. Solve it by separation of variables. For example, if $\frac{dx}{dt} = t/x^2$, $x(0) = 0$, then $x^2 dx = t dt$. Integrate both sides to get $t^2/2 = x^3/3 + c$ so that $x(t) = (3(t^2/2 - c))^{1/3}$. As $x(0) = 0$ we have $c = 0$ and $x(t) = (3t^2/2)^{1/3}$. The linear differential equation $\frac{dx}{dt} = kx$ has the solution $x(t) = e^{kt}x(0)$. For $k > 0$, this means exponential growth. For $k < 0$, exponential decay. A linear system of differential equations is $\frac{dx}{dt} = Ax$. If $x(0) = v$ is an eigenvector with eigenvalue λ , then $x(t)$ is always a multiple of v , say $x(t) = c(t)v$ where $\frac{dv}{dt} = \lambda v$. Thus if $x(0) = c_1 v_1 + \dots + c_n v_n$ writes an initial condition as a sum of eigenvectors, then $x(t) = c_1 e^{\lambda_1 t} v_1 + \dots + c_n e^{\lambda_n t} v_n$ is the closed form solution of the system. It is asymptotically stable, if $x(t) \rightarrow 0$ for all initial conditions $x(0)$. Asymptotic stability holds if and only if $\text{Re}(\lambda_j) < 0$ for all j .