Homework 24: Differential equations I

This homework is due on Friday, April 6, respectively on Tuesday, April 10, 2018.

1 a) Solve the differential equation $\frac{dx}{dt} = t/(5x^4)$,with $x(0) = 2$.
b) Solve the differential equation $\frac{dx}{dt} = 1 + x^2$,with $x(0) = 0$.
c) Solve the differential equation $\frac{dx}{dt} = 1/\cos(x)$,with $x(0) = 0$.

Solution:
Separate variables a) $\int 5x^4 dx = \int t dt + C$ gives $x^5 = t^2/2 + C$ and so $x(t) = (t^2/2 + C)^(1/5)$. If $x(0) = 2$, then $C = 32$. We have $x(t) = (t^2/2 + 32)^{1/5}$.
b) $\arctan(x) = t + c$ gives $x(t) = \tan(t + c)$. As $x(0) = 0$, we have $x(t) = \tan(t)$.
c) $\cos(x)dx = dt$ gives $\sin(x) = t+c$. Fixing the initial condition or $x(t) = \arcsin(t)$.

2 Solve the system

$$\frac{dx}{dt} = Ax, \quad A = \begin{bmatrix} 4 & 9 \\ 7 & 6 \end{bmatrix}$$

with initial condition $x(0) = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$.

Solution:
The eigenvalues are $13, -3$, the eigenvectors are $[1, 1]^T$ and $[9, -7]^T$. Since $[-5, 4]^T = [1, 1]^T + [9, -7]^T$ we have $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{13t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 9 \\ -7 \end{bmatrix}$.

3 a) For which real p, q is the system $\frac{dx}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} x(t)$ stable? b) For which real p is the system $\frac{dx}{dt} = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} x(t)$ stable? c) For which a is the system $\frac{dx}{dt} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} x(t)$ stable?

Solution:
a) The eigenvalues of the matrix are $p + qi, p - qi$, so we obtain coefficient functions $e^{pt}\cos(qt), e^{pt}\sin(qt)$. Thus, the system is stable iff $p < 0$. b) Never. There is always an eigenvalue 1.
c) The eigenvalues are $0, 2a$. The system is never stable as there is always a non-negative eigenvalue.

4 The interaction of two animal species is modeled by the equations

$$\begin{aligned} \frac{dx}{dt} &= 1.5x - 1.2y \\ \frac{dy}{dt} &= 0.8x - 1.4y \end{aligned}$$

a) Interpret the system. Is it a symbiosis, competition or predator-prey?
b) Sketch the phase portrait in the first quadrant.
c) What happens in the long term? Does it depend on the initial population? If so, how?

Solution:
a) This looks like a predator prey relation; $x(t)$ appears to model the prey because its population decreases if the predator $y(t)$ gets bigger. Conversely, the predator grows faster with more prey.
b) There are two eigenvectors in the first quadrant. One is stable, one is unstable.
c) Yes, the outcome depends on the initial condition. It can happen that both strive and increase to infinity or that the pray dies out and the predator population converges to a constant. Once $x = 0$, it stays zero and y will go to zero.

5 A door opens on one side only. A spring mechanism closes the door which forms an angle $\theta(t)$ with the frame. The angular velocity is $\omega(t) = \frac{d\theta}{dt}(t)$. The differential equations are

$$\begin{aligned} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= -2\theta - 3\omega \end{aligned}$$

The first equation is the definition, the second incorporates the force -2θ of the spring and the friction -3ω.
Sketch a phase portrait for the system and use this to answer the question, for which initial conditions, the door slams (reaches $\theta = 0$ with negative ω).

Solution:
The eigenvalues are $-2, -1$. The eigenvectors $[-1, 2]^T$ and $[-1, 1]^T$. The system is stable. Draw the two eigenspaces. If we start below the eigenspace of -2, the trajectory will hit first the line $\theta = 0$, meaning that the door will slam.

Differential Equations I

$\frac{dx}{dt} = f(x)$ is a differential equation. Solve it by separation of variables. For example, if $\frac{dx}{dt} = t/x^2, x(0) = 0$, then $x^2 dx = tdt$. Integrate both sides to get $t^2/2 = x^3/3 + c$ so that $x(t) = (3(t^2/2 - c))^{1/3}$. As $x(0) = 0$ we have $c = 0$ and $x(t) = (3t^2/2)^{1/3}$. The linear differential equation $\frac{dx}{dt} = kx$ has the solution $x(t) = e^{kt}x(0)$. For $k > 0$, this means exponential growth. For $k < 0$, exponential decay. A linear system of differential equations is $\frac{dx}{dt} = Ax$. If $x(0) = v$ is an eigenvector with eigenvalue λ, then $x(t)$ is always a multiple of v, say $x(t) = c(t)v$ where $\frac{dv}{dt} = \lambda v$. Thus if $x(0) = c_1 v_1 + \ldots + c_n v_n$ writes an initial condition as a sum of eigenvectors, then $x(t) = c_1 e^{\lambda_1 t} v_1 + \ldots + c_n e^{\lambda_n t} v_n$ is the closed form solution of the system. It is asymptotically stable, if $x(t) \to 0$ for all initial conditions $x(0)$. Asymptotic stability holds if and only if $\text{Re}(\lambda_j) < 0$ for all j.