Homework 23: Symmetric matrices

This homework is due on Monday, April 2, respectively on Tuesday, April 3, 2018.

1 Give a reason why its true or provide a counterexample.

a) The product of two symmetric matrices is symmetric.

b) The sum of two symmetric matrices is symmetric.

c) The sum of two anti-symmetric matrices is anti-symmetric.

d) The inverse of an invertible symmetric matrix is symmetric.

e) If B is an arbitrary $n \times m$ matrix, then $A = B^T B$ is symmetric.

f) If A is similar to B and A is symmetric, then B is symmetric.

g) $A = SBS^{-1}$ with $S^T S = I_n$, A symmetric $\Rightarrow$ B is symmetric.

h) Every symmetric matrix is diagonalizable.

i) Only the zero matrix is both anti-symmetric and symmetric.

Solution:

a) False, a counterexample is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}$

b) True

c) True

d) True

e) True

f) False, if S is not orthogonal, then this doesn't work. For example, let $A = \begin{bmatrix} 5 & 10 \\ 10 & 32 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 15 & 30 \end{bmatrix} \begin{bmatrix} 5/25 & 0 \\ -1/25 & 5/25 \end{bmatrix}$

g) True: $B^T = B$ implies $A^T = S^T B^T (S^{-1})^T = S^{-1} B S = A$. The same can be done by interchanging A and B and replacing S with S^{-1}. h) True: this is the spectral theorem. i) True: if $A = A^T$ and $A = -A^T$, then $2A = 0$ so that $A = 0$.

2 Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2019 & 2 & 3 & 4 & 5 \\ 2 & 2022 & 6 & 8 & 10 \\ 3 & 6 & 2027 & 12 & 15 \\ 4 & 8 & 12 & 2034 & 20 \\ 5 & 10 & 15 & 20 & 2043 \end{bmatrix}.$$

Solution:
The eigenvalues of $B = A - 2018I_6$ are 0 (with algebraic multiplicity 5). Five of the eigenvectors span the kernel of B and are

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, .$$

The eigenvector to the eigenvalue $1 + 4 + 9 + 16 + 25 = 55$ is

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

because it has to be a vector in the image which is one dimensional and spanned by a column vector. Now add 2018 to get the eigenvalues of 2018,2018,2018,2018, and 2018+55=2073, respectively.

3 a) Find the eigenvalues and orthonormal eigenbasis of $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

b) Find $det(\begin{bmatrix} 7 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 7 \end{bmatrix})$ using eigenvalues

Solution:
a) The eigenvalues are $2, 2, 0, 0$. The eigenvectors lead to the S matrix $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$. We can normalize to that this becomes an orthogonal matrix. b) $B - 5I$ is a matrix with eigenvalues $0, 0, 0, 0$ and 10 (the last number we get by the trace of $B - 5I$). Now A has the eigenvalues $5, 5, 5, 5, 15$. Therefore, the determinant is $5^4 15 = 9375$.

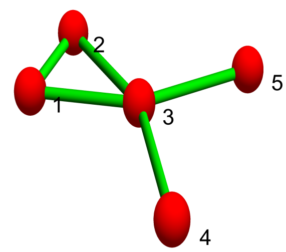
4 Group the matrices which are similar.

$$A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} B=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} C=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} D=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution:
B, C, and D are similar. A is not.

5 Find the eigenvalues and eigenvectors of the Laplacian of the Bunny graph. The Laplacian of a graph with n nodes is the $n \times n$ matrix A which for $i \neq j$ has $A_{ij} = -1$ if i, j are connected and 0 if not. The diagonal entries A_{ii} are chosen so that each row add up to 0.

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$



Solution:
The eigenvalues are $5, 3, 1, 1, 0$. The eigenvectors are $[1, 1, -4, 1, 1]^T, [-1, 1, 0, 0, 0]^T, [-1, -1, 0, 0, 2]^T, [-1, -1, 0, 2, 0]^T, [1, 1$

Symmetric matrices

A is **symmetric** if $A^T = A$ and **anti-symmetric** if $A^T = -A$. Projections or reflections are symmetric. Symmetric matrices appear in physics or statistics: observables like energy, position or momentum matrices are symmetric, correlation matrices are symmetric. In multi-variable calculus the Hessian matrix consisting of the second derivatives is symmetric. The spectral theorem tells that a symmetric matrix has real eigenvalues, that it has an orthonormal eigenbasis and that can be diagonalized as $B = S^{-1}AS$ with an orthogonal matrix S.