## Homework 21: Complex Eigenvalues

This homework is due on Wednesday, March 28, respectively on Thursday, March 29, 2018.

**1** a) For $z = 1 + 4i$. Find $z + z^2 + z^3$.

b) The log of a nonzero complex number $re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$, where $r > 0$ and $0 \le \theta < 2\pi$, is defined as $\log(re^{i\theta}) = \log r + i\theta$. Find the logarithm of $e \cdot i$. ("Ei" is the product of $e$ and $i$. It means "Egg" in German.)

c) Using logarithms we can define $w^z = e^{z \log w}$.
What is $i^i$, the "eye for an eye" number?

**2** a) First use the identity $(\cos(\theta) + i\sin(\theta))^2 = (e^{i\theta})^2 = e^{i2\theta} == \cos(2\theta) + i\sin(2\theta)$ to get the double angle formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

b) Express $\cos(4\theta)$ and $\sin(4\theta)$ as polynomials in $\cos(\theta), \sin(\theta)$.

**3** a) Find all the eigenvalues of the matrix $A = 3Q = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

in the complex plane.

b) Verify that if $\lambda$ is an eigenvalue of $Q$, then $\vec{v} = \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $A$.

**4** Find the eigenvalues/eigenvectors of $A = \begin{bmatrix} 7 & 4 & 2 & 0 & 0 & 1 \\ 1 & 7 & 4 & 2 & 0 & 0 \\ 0 & 1 & 7 & 4 & 2 & 0 \\ 0 & 0 & 1 & 7 & 4 & 2 \\ 2 & 0 & 0 & 1 & 7 & 4 \\ 4 & 2 & 0 & 0 & 1 & 7 \end{bmatrix}$.

Hint: write $A = 7I_6 + 4Q + 2Q^2 + Q^{-1}$?

5 The matrix $A = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.8 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.0 \end{bmatrix}$ is called a Markov matrix: in every column the entries add up to 1, so each column can be interpreted as a probability distribution.

a) Verify that $A^T$ has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue 1.

b) Why is 1 an other eigenvalue of $A$? Find all eigenvalues.
c) Find all eigenvectors of $A$.

## Complex eigenvalues

Complex numbers are of the form $z = a + ib$. Add and multiply as real numbers, keeping in mind $i^2 = -1$. Euler found $\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$ which for $\theta = \pi$ gives $\boxed{e^{i\pi} + 1 = 0}$ (the most beautiful formula in whole of mathematics), as it combines $e, \pi, i, 1$ and $0$. As $e$ is part of analysis and $\pi$ is part of geometry and $0$ is the additive neutral element and $1$ the multiplicative neutral element, this identity combines analysis, geometry and algebra. The Euler identity leads to **de Moivre formulas** like $(\cos(\theta) + i\sin(\theta))^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta) + i\sin(3\theta)$. So that $\cos^3(\theta) - \cos(\theta)\sin^2(\theta) = \cos(3\theta)$ and $\cos^2(\theta)\sin(\theta) - \sin^3(\theta) = \sin(3\theta)$. Eigenvalues of matrices can become complex as the rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ shows which has the eigenvalues $a \pm ib$. The fundamental theorem of algebra assures that the sum of the algebraic multiplicities of all eigenvalues of a $n \times n$ matrix is equal to $n$.