

## Homework 20: Diagonalization

This homework is due on Monday, March 26, respectively on Tuesday, March 27, 2018.

- 1** The **Iodine Heptafluoride molecule**  $IF_7$  has 8 atoms. The

adjacency matrix is 
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
 . a) Verify that the

matrix has the characteristic polynomial  $x^8 - 7x^6$ .

b) Find the eigenvalues of  $A$ .

c) Write down a diagonal matrix  $B$  which is similar to  $A$ .

- 2** Which of the following matrices are diagonalizable? To find out, see whether there is an eigen-basis for  $A$ : a)  $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$  b)

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}, \text{ c) } A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}, \text{ d) } A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

- 3** Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hint: Compute  $A^2$  and  $B^2$  and use that if  $A$  and  $B$  are similar then  $A^2$  and  $B^2$  are similar.

4 Find  $f(A) = A^6 + A^4 + A$  for  $A = \begin{bmatrix} 6 & -2 \\ 3 & 1 \end{bmatrix}$  by diagonalization: find a matrix  $B = S^{-1}AS$  which is diagonalizable, then compute  $f(B)$  and then transform back  $f(A) = Sf(B)S^{-1}$ .

5 Let  $A$  be a nonzero  $3 \times 3$  matrix for which  $A^2 = 0$ . We know that the image is a subspace of the kernel of  $A$ . a) Verify that the image has dimension 1 and the kernel dimension 2.

Pick  $v_1$  in the image of  $A$  and write  $v_1 = Av_2$ . Let  $v_3$  be a vector in the kernel which is not a multiple of  $v_1$ .

b) Verify that  $\mathcal{B} = \{v_1, v_2, v_3\}$  is a basis. Find the matrix  $B$  describing the matrix  $A$  in that basis.

c) The matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$  is not diagonalizable. Use b) to

find  $S$  such that  $S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

## Diagonalization

If  $A$  is similar to a diagonal matrix  $B$ , then  $A$  is called **diagonalizable**. In that case the coordinate transformation  $S$  has the eigenvectors of  $A$  as columns. A key result is that every  $n \times n$  matrix which has  $n$  different eigenvalues is diagonalizable. The reason is that the eigenvectors form then an eigenbasis. If  $A$  is diagonalizable with diagonal matrix  $B = S^{-1}AS$  one can define  $f(A)$  for any function  $f$  by  $f(A) = Sf(B)S^{-1}$  where  $f$  is applied to each diagonal entry of  $B$ . For example  $\sin\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$  is  $S \begin{bmatrix} \sin(0) & 0 \\ 0 & \sin(2) \end{bmatrix} S^{-1}$  with  $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$  which is  $\begin{bmatrix} \frac{\sin(2)}{2} & -\frac{\sin(2)}{2} \\ -\frac{\sin(2)}{2} & \frac{\sin(2)}{2} \end{bmatrix}$ .