Homework 20: Diagonalization

This homework is due on Monday, March 26, respectively on Tuesday, March 27, 2018.

1 The **Iodine Heptafluoride molecule** IF_7 has 8 atoms. The adjacency matrix is $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. a) Verify that the matrix has the characteristic polynomial $x^8 - 7x^6$.

b) Find the eigenvalues of A.

c) Write down a diagonal matrix B which is similar to A.

Solution:

a) To compute the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_8)$, we can either do a Laplace expansion or see that there are 8 patterns and get λ^8 for the diagonal pattern and 7 patterns λ^6 with three diagonal entries.

b) The characteristic polynomial $f_A(\lambda)$ factors as $x^6(x^2 - 7)$.

c) The eigenvalues are $\lambda = 0$ with multiplicity 6 The eigenvalues are 0 (algebraic multiplicity 6) and $\pm\sqrt{7}$ each with multiplicity 1. The diagonal matrix has diagonal entries $[0, 0, 0, 0, 0, 0, \sqrt{7}, -\sqrt{7}]$

$$\begin{bmatrix} -\sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 Which of the following matrices are diagonalizable? To find out, see whether there is an eigen-basis for A: a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ b) $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$, c) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, d) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$,

Solution:
a) The matrix has two different eigenvalues $\lambda = 5, -1$, and therefore is diagonalizable: we have an eigenvector for each eigenvalue and they are not parallel so that they form a basis.
b) There are two different eigenvalues, $\lambda = \pm\sqrt{13}$, therefore the matrix is diagonalizable.
c) The matrix is not diagonalizable: the geometric multiplicity of the lone eigenvalue $\lambda = 2$ is 1.
d) The matrix is already diagonal. It is therefore diagonalizable.

3 Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hint: Compute A^2 and B^2 and use that if A and B are similar then A^2 and B^2 are similar.

Solution:
While A, B have the same eigenvalues $0, 0, 0, 0$ and geometric multiplicity 2 for the eigenvalue 0, the matrices are still not similar. If they were, then their squares would be similar as well. However, A^2 and B^2 have different geometric multiplicities for the eigenvalue 0, and therefore cannot be similar.

4 Find $f(A) = A^6 + A^4 + A$ for $A = \begin{bmatrix} 6 & -2 \\ 3 & 1 \end{bmatrix}$ by diagonalization: find a matrix $B = S^{-1}AS$ which is diagonalizable, then compute $f(B)$ and then transform back $f(A) = Sf(B)S^{-1}$.

Solution:
We compute the eigenvalues, and obtain $\lambda = 4, 3$. This tells us that the matrix A is similar to a diagonal matrix $B = Diag(4, 3)$. We can compute $f(B) = B^6 + B^4 + B$ by just applying f to the diagonal elements. We have

$$f(B) = \begin{bmatrix} 4^6 + 4^4 + 4^1 & 0 \\ 0 & 3^6 + 3^4 + 3^1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

We now conjugate to get $A = SBS^{-1}$. The conjugating matrix S has the eigenvectors in the columns. The inverse is needed as well:

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, S^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} .$$

Now, $f(A) = Sf(B)S^{-1} = \begin{bmatrix} 11442 & -7086 \\ 10629 & -6273 \end{bmatrix}$.

5 Let A be a nonzero 3×3 matrix for which $A^2 = 0$. We know that the image is a subspace of the kernel of A. a) Verify that the image has dimension 1 and the kernel dimension 2.
Pick v_1 in the image of A and write $v_1 = Av_2$. Let v_3 be a vector in the kernel which is not a multiple of v_1.
b) Verify that $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis. Find the matrix B describing the matrix A in that basis.
c) The matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$ is not diagonalizable. Use b) to find S such that $S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution:

a) If x is in the image, then $x = Ay$ for some nonzero y. But now $Ax = A^2y = 0$, so that x is in the kernel of A. Because the kernel is at least one dimensional and by the rank-nullity theorem, the dimension of the kernel and image add up to 3, we must have that the kernel is two dimensional and the image one dimensional. (The other possibility that the kernel is one dimensional and the image is 2 dimensional is not possible because the image is contained in the kernel).

b) If v_1 is in the image and $v_1 = Av_2$, then v_2 can not be in the plane which makes up the kernel because otherwise, $Av_2 = 0$. So, v_2 sticks out from the plane. If we pick up a third vector v_3 in the kernel so that v_1, v_3 span the plane, then we have a basis, as v_2 is not in the plane. Lets look at the matrix in the new basis: Since $Av_1 = 0, Av_3 = 0$, the first and third column of the matrix B is zero. Because $Av_2 = v_1 = 1v_1 + 0v_2 + 0v_3$, the second column is the vector e_1.

c) The matrix A is just an example illustrating this. Take $v_1 = [1, 2, 3]^T$ (the first column), then $v_2 = e_1 = [1, 0, 0]^T$. We only need to find an other vector in the kernel. $[1, 0, -1]^T$ does. Now we have the matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

for which we know that $B = S^{-1}AS$ has the desired form.

Diagonalization

If A is similar to a diagonal matrix B, then A is called **diagonalizable**. In that case the coordinate transformation S has the eigenvectors of A as columns. A key result is that every $n \times n$ matrix which has n different eigenvalues is diagonalizable. The reason is that the eigenvectors form then an eigenbasis. If A is diagonalizable with diagonal matrix $B = S^{-1}AS$ one can define $f(A)$ for any function f by $f(A) = Sf(B)S^{-1}$ where f is applied to each diagonal entry of B. For example $\sin(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$ is $S \begin{bmatrix} \sin(0) & 0 \\ 0 & \sin(2) \end{bmatrix} S^{-1}$ with $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ which is $\begin{bmatrix} \frac{\sin(2)}{2} & -\frac{\sin(2)}{2} \\ -\frac{\sin(2)}{2} & \frac{\sin(2)}{2} \end{bmatrix}$.