Homework 19: Eigenspaces

This homework is due on Friday, March 23, respectively on Tuesday, March 27, 2018.

1 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$.
What are the algebraic and geometric multiplicities? As a hint, we tell you that the eigenvectors are
$[-2,0,1,0,1]^T, [-2,1,0,1,0]^T, [0,0,-1,0,1]^T$,
$[0,-1,0,1,0]^T, [1,1,1,1,1]^T$. Now find the characteristic polynomial of A.

Solution:
The eigenvalues are $5,5,3,3,0$. The algebraic multiplicity of 3 and 5 is 2. The algebraic multiplicity of 0 is 1. The geometric multiplies are equal to the algebraic multiplicities.

2 Assume that a 2×2 matrix has trace 9 and determinant 14. Find its eigenvalues and find a non-diagonal matrix which realizes the situation.

Solution:
We have $\lambda_1 + \lambda_2 = 9$ and $\lambda_1\lambda_2 = 14$. By inspection, we see that $\lambda = 2, 7$. We can realize it as a triangular matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}.$$

3 a) Verify that for any $n \times n$ matrix A, the matrix A and A^T have the same eigenvalues.
b) Assume A is invertible. What is the relation between the eigenvalues of A and A^{-1}?

Solution:
a) The characteristic polynomial of A^T is $\det(A^T - \lambda 1) = \det((A-\lambda 1)^T) = \det(A-\lambda 1)$, which is precisely the characteristic polynomial of A. Thus, the roots (which are the eigenvalues) must be the same.
b) The eigenvalues of A^{-1} are the inverses of the eigenvalues of A.

4 This is a classic problem from Otto Bretscher. The vector $A^n b$ gives pollution levels in the Silvaplana, Sils and St Moritz lake n weeks after an oil spill. The matrix is $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ and $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is the initial pollution level. Find a closed form solution for the pollution after n weeks.

Solution:

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Show[
  {ListPlot[{100., 70., 49., 34.3, 24.01, 16.807, 11.7649, 8.23543, 5.7648, 4.03536, 2.82475},
      PlotJoined→True, DisplayFunction→Identity],
    ListPlot[{0., 10., 1., 4.3, 0.85, 1.891, 0.5461, 0.84883, 0.314245, 0.387933, 0.170776},
      PlotJoined→True, DisplayFunction→Identity],
    ListPlot[{0., 0., 2., 1.8, 2.3, 2.01, 1.9862, 1.69818, 1.52831, 1.2855, 1.10598},
      PlotJoined→True, DisplayFunction→Identity]},
  PlotRange→All, DisplayFunction→Identity]
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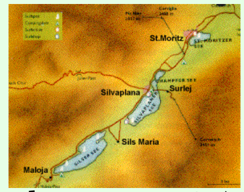

The eigenvalues are the diagonal elements $0.7, 0.6, 0.8$. There is an eigenvector $e_3 = v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for the eigenvalue $\lambda_3 = 0.8$.

There is an eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ for the eigenvalue $\lambda_2 = 0.6$.

There is an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ for the eigenvalue $\lambda_1 = 0.7$.

From this, we know $A^n v_1$, $A^n v_2$ and $A^n v_3$ explicitly.
From the initial conditions, we also know $b = 100 \cdot e_1 = 100(v_1 - v_2 + v_3)$. So we have

$$\begin{aligned} A^n(b) &= 100A^n(v_1 - v_2 + v_3) = 100(\lambda_1^n v_1 - \lambda_2^n v_2 + \lambda_3^n v_3) \\ &= 100\left(0.7^n \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - 0.6^n \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 0.8^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{aligned}$$

5 a) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 6 & 6 & 0 \\ 1 & 14 & 36 & 24 \end{bmatrix}.$$

b) Find the eigenvectors of A^3, where A is the previous matrix.
c) Find the eigenvectors of $(A^T)^{-1}$, where A is the previous matrix.

Solution:
a) The eigenvalues are the diagonal elements. The eigenvectors are $[[0,0,0,1],[0,0,-1,2],[0,22,-33,40],[-1,1,-1,1]$.
b) The eigenvalues of A^3 are λ^3 for each eigenvalue λ of A, with the same eigenvectors as before.
c) It is $1/\lambda$ if λ is an eigenvalue. For the eigenvectors, we have to do the computation again since the eigenvectors of A^T are different. We get the eigenvectors $[[2,13,22,11],[1,3,2,0],[1,1,0,0],[1,0,0,0]]$.

Eigenspaces

A nonzero vector $\vec{v}$ is called an **eigenvector**, if $Av = \lambda v$ for some λ. The set of eigenvectors is called the **eigenspace** E_λ. It is the kernel of $A - \lambda I_n$. The dimension of the eigenspace is called the **geometric multiplicity** of λ. There is a general result which tells that the geometric multiplicity of λ is always smaller or equal to the algebraic multiplicity.
Recall that A is similar to B if there exists an invertible S such that $B = S^{-1}AS$. If A and B are similar, then they have the same characteristic polynomial, the same eigenvalues and algebraic multiplicities as well as the same geometric multiplicities. Similar matrices also have the same trace $\text{tr}(A) = \lambda_1 + ... + \lambda_n$ as well as determinant $\det(A) = \lambda_1 \cdots \lambda_n$. These formulas hold in general if we allow the eigenvalues λ_i to be complex. More on complex eigenvalues next week.