

Homework 18: Discrete Dynamical systems

This homework is due on Wednesday, March 21, respectively on Thursday, March 22, 2018.

- 1 a) Show that $\begin{bmatrix} 4 & 12 \\ 3 & 13 \end{bmatrix}$ has the eigenvalues 16 and 1. Find the corresponding eigenvectors v and w . b) What is $A^5v + A^5w$?
- 2 a) The matrix $A = \begin{bmatrix} 12 & 5 \\ 5 & -12 \end{bmatrix}$ is a reflection dilation. Use geometric insight to find the eigenvalues and eigenvectors of A .
- b) The matrix $B = \begin{bmatrix} 6 & 0 & -8 \\ 0 & 4 & 0 \\ 8 & 0 & 6 \end{bmatrix}$ is a rotation dilation on the xz -plane and a dilation in the y -axes. Use this to find an eigenvector and eigenvalue.
- 3 A Lilac bush has $n(t)$ new branches and $o(t)$ old branches at the beginning of each year t . During the year, each old branch will grow two new branches and remain old and every new branch will become a old branch.
- a) Find the matrix A such that $\begin{bmatrix} n(t+1) \\ o(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ o(t) \end{bmatrix}$.
- b) Verify that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors. Find the eigenvalues.
- c) Find closed formulas for $n(t)$, $o(t)$ if the initial condition $\begin{bmatrix} n(0) \\ o(0) \end{bmatrix} = c_1v_1 + c_2v_2$ are given. If you prefer to work with an example, take the initial condition $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

4 a) Find the characteristic polynomial of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and

determine its roots.

b) What are the eigenvalues and eigenvectors of the projection $P(x, y, z) = (x, y, 0)$ from space to the xy -plane?

5 a) Find the characteristic polynomial of 5×5 matrix for which the entries are $A_{kl} = k + l$ if $k \geq l$ and $A_{kl} = 0$ if $k < l$.

b) Why does every 11×11 matrix have a real eigenvalue.

c) Find a concrete 4×4 matrix which has no real eigenvalue.

Eigenvalues

A **nonzero** vector v is an **eigenvector** of A , if $Av = \lambda v$ for some real number λ called **eigenvalue**. A basis \mathcal{B} consisting of eigenvectors of A is called an **eigenbasis**. Eigenvalues λ_j and vectors v_j help to solve **discrete dynamical systems** $x \rightarrow Ax$, where we want to find closed formulas for the trajectories $A^t x$: write an initial vector x as a sum of eigenvectors $x = c_1 v_1 + \dots + c_n v_n$, then get $A^t x = c_1 \lambda_1^t v_1 + \dots + c_n \lambda_n^t v_n$. One can find eigenvalues as roots of the **characteristic polynomial** $f_A(\lambda) = \det(A - \lambda I_n)$. It is a polynomial of degree n of the form

$$f_A(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \dots + \det(A).$$

The algebraic multiplicity of an eigenvalue is the multiplicity of the root. The matrix

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for example has the characteristic polynomial

$$f_A(\lambda) = \det \left(\begin{bmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{bmatrix} \right)$$

which is $-\lambda^3 + 3\lambda^2 = \lambda^2(3 - \lambda)$ showing that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue of multiplicity 1.