Homework 18: Discrete Dynamical systems

This homework is due on Wednesday, March 21, respectively on Thursday, March 22, 2018.

1 a) Show that $\begin{bmatrix} 4 & 12 \\ 3 & 13 \end{bmatrix}$ has the eigenvalues 16 and 1. Find the corresponding eigenvectors v and w. b) What is $A^5v + A^5w$?

Solution:
a) We can solve the equation $(A - 16)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -12 & 12 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ to find an eigenvector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similarly, to find a second eigenvector $w = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$. b) $A^5v + A^5w = 16^5v + w = [16^5, 16^5]^T + [-4, 1]^T$.

2 a) The matrix $A = \begin{bmatrix} 12 & 5 \\ 5 & -12 \end{bmatrix}$ is a reflection dilation. Use geometric insight to find the eigenvalues and eigenvectors of A.
b) The matrix $B = \begin{bmatrix} 6 & 0 & -8 \\ 0 & 4 & 0 \\ 8 & 0 & 6 \end{bmatrix}$ is a rotation dilation on the xz-plane and a dilation in the y-axes. Use this to find an eigenvector and eigenvalue.

Solution:
a) The eigenvalues of a reflection at a line are always $1, -1$. The eigenvalue 1 belongs to the eigenvector which is in the line. The eigenvalue -1 belongs to the vector perpendicular to it. In the case of a rotation-dilation, the matrix also scales by a factor 13, therefore the eigenvalues of A are $13, -13$. The eigenvectors are vectors in the line of reflection and in the line perpendicular to that. One way to get the vector of the line of reflection geometrically is to look at the parallelogram with vertices $[0,0]^T, [13,0]^T, A[1,0]^T, [13,0]^T + A[1,0]^T = [25,5]^T$. The vector $[5,1]^T$ is therefore an eigenvector to the eigenvalue 13. The vector $[-1,5]$ is perpendicular and so an eigenvector to the eigenvalue -13.
b) The only real eigenvalue is 4, with eigenvector $[0,1,0]$. The rotation part in the xz plane does not have any real eigenvalue as no nonzero vector is fixed.

3 A Lilac bush has $n(t)$ new branches and $o(t)$ old branches at the beginning of each year t. During the year, each old branch will grow two new branches and remain old and every new branch will become a old branch.
a) Find the matrix A such that $\begin{bmatrix} n(t+1) \\ o(t+1) \end{bmatrix} = A \begin{bmatrix} n(t) \\ o(t) \end{bmatrix}$.
b) Verify that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ are eigenvectors. Find the eigenvalues.
c) Find closed formulas for $n(t), o(t)$ if the initial condition $\begin{bmatrix} n(0) \\ o(0) \end{bmatrix} = c_1 v_1 + c_2 v_2$ are given. If you prefer to work with an example, take the initial condition $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Solution:
Since the vector e_1 which represents one new branch and no old branch goes into e_2 which represents one old branch and because the vector e_2 which represents one old branch becomes $2e_1 + e_2$, we can see the column vectors of the matrix A and get

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} .$$

The eigenvalues can be obtained by directly plugging in the vectors. We have $A\vec{v}_1 = 2\vec{v}_1$ and $A\vec{v}_2 = (-1)\vec{v}_2$. To get the solution for a concrete initial condition, write $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 = 2\vec{v}_1 + \vec{v}_2$. Now $A^t\vec{v} = c_1 2^t\vec{v}_1 + c_2(-1)^t\vec{v}_2$, so we see that

$$\begin{bmatrix} n(t) \\ o(t) \end{bmatrix} = \begin{bmatrix} 2^{t+1} + (-1)^t 2 \\ 2^{t+1} - (-1)^t \end{bmatrix} .$$

4 a) Find the characteristic polynomial of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and determine its roots.

b) What are the eigenvalues and eigenvectors of the projection $P(x, y, z) = (x, y, 0)$ from space to the xy-plane?

Solution:

a) We have $\det\left(\begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}\right) = -x^3+6x^2-10x+4$.

The characteristic polynomial has the roots $\lambda = 2, 2 \pm \sqrt{2}$.

b) We can solve this problem by writing down the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and determining the characteristic polynomial $(1-\lambda)^2(-\lambda)$ which has the roots 1 and 0. The eigenvectors to the eigenvalue 1 can be obtained by computing the kernel of $A - 1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvector to the eigenvalue 0 generates the kernel of A.

This tells us that every vector in the xy-plane is an eigenvector for the eigenvalue 1, and the vector e_3 in the z-axes is the eigenvector for the eigenvalue 0.

5 a) Find the characteristic polynomial of 5×5 matrix for which the entries are $A_{kl} = k + l$ if $k \geq l$ and $A_{kl} = 0$ if $k < l$.

b) Why does every 11×11 matrix have a real eigenvalue.

c) Find a concrete 4×4 matrix which has no real eigenvalue.

Solution:
a) The characteristic polynomial is $(2-\lambda)(4-\lambda)(6-\lambda)(8-\lambda)(10-\lambda)$.
b) The characteristic polynomial is an odd degree polynomial, which has a dominant term $(-\lambda)^{11}$ which goes to $-\infty$ at ∞ and to ∞ for $\lambda \to -\infty$. By the intermediate value theorem there is a root.
c) Take $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. which has the characteristic polynomial $\lambda^2 + 1$ which has no real roots. Now, for any n, we can construct a $2n \times 2n$ matrix B with no real roots by placing n copies of A along the diagonal (more precisely, each submatrix $(B_{ij})_{i,j \in \{2k-1,2k\}}$, for k in $\{1, \ldots, n\}$ is A). For $n = 3$, we obtain the 4×4 matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Eigenvalues

A **nonzero** vector v is an **eigenvector** of A, if $Av = \lambda v$ for some real number λ called **eigenvalue**. A basis $\mathcal{B}$ consisting of eigenvectors of A is called an **eigenbasis**. Eigenvalues λ_j and vectors v_j help to solve **discrete dynamical systems** $x \to Ax$, where we want to find closed formulas for the trajectories $A^t x$: write an initial vector x as a sum of eigenvectors $x = c_1 v_1 + ... + c_n v_n$, then get $A^t x = c_1 \lambda_1^t v_1 + ... + c_n \lambda_n^t v_n$. One can find eigenvalues as roots of the **characteristic polynomial** $f_A(\lambda) = \det(A - \lambda I_n)$. It is a polynomial of degree n of the form

$$f_A(\lambda) = (-\lambda)^n + \mathrm{tr}(A)(-\lambda)^{n-1} + ... + \det(A) \; .$$

The algebraic multiplicity of an eigenvalue is the multiplicity of the root. The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for example has the characteristic polynomial

$$f_A(\lambda) = \det(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix})$$

which is $-\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda)$ showing that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue of multiplicity 1.