Homework 17: Determinants II

This homework is due on Monday, March 19, respectively on Tuesday, March 20, 2018. Its a good idea to finish this before spring break!

1 a) We find here the determinant of the 5×5 matrix A for which the entry A_{km} is $\phi(k+m)$, where ϕ is the Euler totient function giving the number of positive integers less than n that are co-prime to n. Use row reduction to find the determinant:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix} .$$

b) Find the determinant of the GCD matrix of size 500×500. The result has only 1009 digits. What is the first digit? Excessive homework? No! You have all spring break ...

Hint: use a machine:

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M=500; Det[Table[GCD[n,k],{n,M},{k,M}]];
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Solution:

a) 16

b) Run the code. It is 3.

2 a) Find the determinant of

$$B = \begin{bmatrix} 4 & 6 & 0 & 0 & 0 & 0 \\ 4 & 3 & 3 & 0 & 0 & 0 \\ 4 & 4 & 4 & 3 & 0 & 0 \\ 4 & 4 & 4 & 4 & 3 & 0 \\ 4 & 4 & 4 & 4 & 4 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{bmatrix} .$$

b) Find the determinant of $9B$.

Solution:
a) Row reducing by subtracting each row by the next upper row gives an upper triangular matrix We know that its determinant is simply the product of the diagonal elements which gives $\boxed{\det(B) = -24}$.
b) $\det(7B) = 9^6 \det(B) = 9^6 \cdot 24 = -12754584$.

3 a) Find the determinant of

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 6 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 2 & 4 & 4 \end{bmatrix}$$

b) Find the determinant of A^5.

Solution:
a) This is a partitioned matrix. The determinants of the two 3×3 diagonal blocks are -18 and 40. The product is -720.
b) We have $\det(A^5) = \det(A)^5 = -720^5$.

4 Argue geometrically why the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

has maximal absolute determinant $|\det(A)|$ among all matrices with entries in $\{-1, 1\}$.

Solution:
The determinant is the volume of a parallelepiped spanned by the column vectors. In this case, the columns of the matrix are perpendicular and have length 2, yielding volume 16. In general, the volume of a parallelepiped can be at most the product of its side lengths, which will always be 16 for such matrices.

5 a) Find A, B such that $\det(A - B) \neq \det(A) - \det(B)$.
b) What values can an orthogonal matrix have?

c) Verify that $|\det(A)|$ only depends on R if $A = QR$ is the QR factorization.

Solution:
a) Take the identity 2×2 matrix for A and $B = A$. Now $A + B = 0$ and $\det(A) + \det(B) \neq 0$.
b) Write $A^T A = 1$ and take the determinant on both sides. Using the product and transpose property we see

$$\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A)$$

. But, $\det(A^T A) = \det(1) = 1$, so we have $\det(A)^2 = 1$. Thus, $\det(A) \in \{-1, 1\}$. Note that an othogonal matrix has orthonormal columns but that the term "orthonormal matrix" does not exist.
c) We once again use the product property, to obtain $\det(A) = \det(Q)\det(R) = \pm\det(R)$. Therefore, $|\det(A)| = |\det(R)|$.

Determinants II

Determinants can be computed using row reduction: If during row reduction m swapping operations have occurred and the scaling factors are $c_1, ..., c_k$, then

$$\det(A) = \frac{(-1)^m}{c_1 \cdots c_k} \det(\text{rref}(A))$$

Here are some more properties:

- $|\det(A)|$ is the volume of a parallel epiped
- $\det(AB) = \det(A)\det(B)$
- $\det(A^T) = \det(A)$
- $\det(A^n) = (\det(A))^n$
- $\det(A^{-1}) = 1/\det(A)$